Deviations from linear beam dynamics in the form of perturbations and aberrations play an important role in accelerator physics. Beam parameters, quality, and stability are determined by our ability to correct and control such perturbations. Hamiltonian formulation of nonlinear beam dynamics allows us to study, understand, and quantify the effects of geometric and chromatic aberrations in higher order than discussed so far. Based on this understanding we may develop correction mechanisms to achieve more and more sophisticated beam performance. We will first discuss higher order beam dynamics as an extension to the linear matrix formulation followed by specific discussions on aberrations. Finally, we develop the Hamiltonian perturbation theory for particle beam dynamics in accelerator systems.

14.1 Higher Order Beam Dynamics

Chromatic and geometric aberrations appear specifically in strong focusing transport systems designed to preserve carefully prepared beam characteristics. As a consequence of correcting chromatic aberrations by sextupole magnets, nonlinear geometric aberrations are introduced. The effects of both types of aberrations on beam stability must be discussed in some detail. Based on quantitative expressions for aberrations, we will be able to determine criteria for stability of a particle beam.

14.1.1 Multipole Errors

The general equations of motion (3.75), (3.76) exhibit an abundance of driving terms which depend on second or higher order transverse particle coordinates \((x, x', y, y')\) or linear and higher order momentum errors \(\delta\). Magnet alignment and field errors add another multiplicity to these perturbation terms. Although the designers of accelerator lattices and beam guidance magnets take great care to minimize undesired field components and avoid focusing systems...
that can lead to large transverse particle deviations from the reference orbit, we cannot completely ignore such perturbation terms.

In previous sections we have discussed the effect of some of these terms and have derived among other effects such basic beam dynamics features as the dispersion function, orbit distortions, chromaticity, and tune shifts as a consequence of particle momentum errors or magnet alignment and field errors. More general tools are required to determine the effect of any arbitrary driving term on the particle trajectories. In developing such tools we will assume a careful design of the accelerator under study in layout and components so that the driving terms on the r.h.s. of (3.75), (3.76) can be treated truly as perturbations. This may not be appropriate in all circumstances in which cases numerical methods need to be applied. For the vast majority of accelerator physics applications it is, however, appropriate to treat these higher order terms as perturbations.

This assumption simplifies greatly the mathematical complexity. Foremost, we can still assume that the general equations of motion are linear differential equations. We may therefore continue to treat every perturbation term separately as we have done so before and use the unperturbed solutions for the amplitude factors in the perturbation terms. The perturbations are reduced to functions of the location $z$ along the beam line and the relative momentum error $\delta$ only and such differential equations can be solved analytically as we will see. Summing all solutions for the individual perturbations finally leads to the composite solution of the equation of motion in the approximation of small errors.

The differential equations of motion (3.75), (3.76) can be expressed in a short form by

$$u'' + K(z) u = \sum_{\mu, \nu, \sigma, \rho, \tau \geq 0} p_{\mu\nu\sigma\rho\tau}(z) x^\mu x'^\nu y^\sigma y'^\rho \delta^\tau,$$

(14.1)

where $u = x$ or $u = y$ and the quantities $p_{\mu\nu\sigma\rho\tau}(z)$ represent the coefficients of perturbation terms. The same form of equation can be used for the vertical plane but we will restrict the discussion to only one plane neglecting coupling effects.

Some of the perturbation terms $p_{\mu\nu\sigma\rho\tau}$ can be related to aberrations known from geometrical light optics. Linear particle beam dynamics and Gaussian geometric light optics work only for paraxial beams where the light rays or particle trajectories are close to the optical axis or reference path. Large deviations in amplitude, as well as fast variations of amplitudes or large slopes, create aberrations in the imaging process leading to distortions of the image known as spherical aberrations, coma, distortions, curvature, and astigmatism. While corrections of such aberrations are desired, the means to achieve corrections in particle beam dynamics are different from those used in light optics. Much of the theory of particle beam dynamics is devoted to diagnose the effects of aberrations on particle beams and to develop and apply such corrections.