Single Particle Dynamics

The general equations of motion, characterized by an abundance of perturbation terms on the right-hand side of, for example, (3.75), (3.76) have been derived in the previous chapter. If these perturbation terms were allowed to become significant in real beam transport systems, we would face almost insurmountable mathematical problems trying to describe the motion of charged particles in a general way. For practical mathematical reasons, it is therefore important to design components for particle beam transport systems such that undesired terms appear only as small perturbations. With a careful design of beam guidance magnets and accurate alignment of these magnets we can indeed achieve this goal.

Most of the perturbation terms are valid solutions of the Laplace equation describing higher order fields components. Virtually all these terms can be minimized to the level of perturbations by a proper design of beam transport magnets. Specifically, we will see that the basic goals of beam dynamics can be achieved by using only two types of magnets, bending magnets and quadrupoles, which sometimes are combined into one magnet. Beam transport systems, based on only these two lowest order magnent types, are called linear systems and the resulting theory of particle dynamics in the presence of only such magnets is referred to as linear beam dynamics or linear beam optics.

In addition to the higher order magnetic field components, we also find purely kinematic terms in the equations of motion due to large amplitudes or due to the use of curvilinear coordinates. Some of these terms are generally very small for particle trajectories which stay close to the reference path such that divergences are small, $x' \ll 1$ and $y' \ll 1$. The lowest order kinematic effects resulting from the use of a curvilinear coordinate system, however, cannot generally be considered small perturbations. One important consequence of this choice for the coordinate system is that the natural bending magnet is a sector magnet which has very different beam dynamics properties than a rectangular magnet which would be the natural magnet type for a Cartesian coordinate system. While a specific choice of a coordinate system will not change the physics, we must expect that some features are expressed easily
or in a more complicated way in one or the other coordinate system. We have chosen to use the curvilinear system because it follows the ideal path of the particle beam and offers a simple direct way to express particle trajectories deviating from the ideal path. In a fixed Cartesian coordinate system, we would have to deal with geometric expressions relating the points along the ideal path to an arbitrary reference point. The difference becomes evident for a simple trajectory like a circle of radius $r$ and center at $(x_0, y_0)$ which in a fixed orthogonal coordinate system would be expressed by $(x-x_0)^2 + (y-y_0)^2 = r^2$. In the curvilinear coordinate system this equation reduces to the simple identity $x(z) = 0$.

### 4.1 Linear Beam Transport Systems

The theory of beam dynamics based on quadrupole magnets for focusing is called strong focusing beam dynamics in contrast to the case of weak focusing. Weak focusing systems utilize the focusing of sector magnets in combination with a small gradient in the bending magnet profile. Such focusing is utilized in circular accelerators like betatrons or some cyclotrons and the first generation of synchrotrons. The invention of strong focusing by Christofilos [18] and independently by Courant et al. [19] changed quickly the way focusing arrangements for large particle accelerators are determined. One of the main attractions for this kind of focusing was the ability to greatly reduce the magnet aperture needed for the particle beam since the stronger focusing confines the particles to a much smaller cross section compared to weak focusing. A wealth of publications and notes have been written during the 1950s to determine and understand the intricacies of strong focusing, especially the rising problems of alignment and field tolerances as well as those of resonances. Particle stability conditions from a mathematical point of view have been investigated by Moser [36].

Extensive mathematical tools are available to determine the characteristics of linear particle motion. In this chapter, we will discuss the theory of linear charged particle beam dynamics and apply it to the development of beam transport systems, the characterization of particle beams, and to the derivation of beam stability criteria.

The bending and focusing function may be performed either in separate magnets or be combined within a synchrotron magnet. The arrangement of magnets in a beam transport system, called the magnet lattice, is often referred to as a separated function or combined function lattice depending on whether the lattice makes use of separate dipole and quadrupole magnets or uses combined function magnets, respectively.

Linear equations of motion can be extracted from (3.75), (3.76) to treat beam dynamics in first or linear approximation. For simplicity and without restricting generality we assume the bending of the beam to occur only in one plane, the $x$-plane. The linear magnetic fields for bending and quadrupole