8.1 INTRODUCTION

Forces applied to solids cause deformation, and forces applied to liquids cause flow.

(Fung, 1977)

Primary volcanic landforms are created by the ascent and eruption of magma. The ascending magma displaces and interacts with surrounding rock and fluids as it creates new pathways, flows through cracks or conduits, vesiculates, and accumulates in underground reservoirs. The formation of new pathways and pressure changes within existing conduits and reservoirs stress and deform the surrounding rock. Eruption products load the crust. The pattern and rate of surface deformation around volcanoes reflect the tectonic and volcanic processes transmitted to the surface through the mechanical properties of the crust.

Mathematical models, based on solid and fluid mechanics, have been developed to approximate deformation from tectonic and volcanic activity. Knowledge of the concepts and limitations of continuum mechanics is helpful to understanding this chapter. The models predict surface deformation from forces acting, or displacements occurring, within the Earth. These subterranean forces or displacements are referred to as sources of deformation. Quantitative estimates of their location, geometry, and dynamics are inferred by comparing or fitting surface observations to the predictions from these idealized mathematical models.

We do not derive the equations that relate forces or displacements at the source to deformation at the surface, but we do provide an overview of the methods and references for such derivations. These source models are mathematical abstractions and, as a result, this chapter is filled with equations that may be daunting, but we use plots and tables to describe important characteristics of the predicted deformation.

Volcanic deformation sources include inflating, deflating, and growing bodies of various shapes and sizes, which are collectively known as volumetric sources. The opening or closing of a cavity or crack is distinct from the typical tectonic source, such as a strike-slip or dip-slip fault, where the two sides of a fault slide by one another. Of course, there are composite sources that include both tensile and shear movements, such as a leaky transform fault. Volumetric sources grow and shrink through the movement of fluids and, in some cases, include both sources and sinks. For example, magma filling a growing dike is drawn from (deflates) an adjacent magma chamber.

Observed surface deformation can be fit to the predictions of the source models. The modeling of surface deformation, however, does not provide a unique description of the source causing the deformation. Even with a perfect description of the surface deformation, we could find many different ways to account for it. Model assumptions, simplifications, and data uncertainty further complicate interpretation. Nevertheless, much can be learned from non-unique modeling of sparse and imprecise data.

We are interested in predicting or fitting geodetic data: station displacements, line length changes, tilt, and strain. We limit our discussion to the slow static changes that occur over long periods of time and permanent offsets associated with volcanic or tectonic events. We do not discuss oscillatory, high-frequency ground motions, such as the dynamic strains that accompany earthquakes, even though they sometimes excite volcanic systems. Earth scientists do not like to label any change as static, so they
often refer to these very low frequency ground movements as quasi-static ground deformation.

This chapter is more elementary than a previous discussion on modeling ground deformation in volcanic areas by De Natale and Pingue (1996), although there is some overlap. Both chapters include a short introduction to the theory of elementary strain sources in an elastic medium, a summary of spheroidal pressure sources, and a discussion of the ambiguities inherent in modeling surface deformation. DeNatale and Pingue extend their discussion to modifications needed to make the elastic half-space models more realistic. These include inversion techniques for non-uniform pressure distributions and crack openings, and the effects of inhomogeneity, plastic rheologies, and structural discontinuities. The simplifications that make analytical models tractable can, particularly in the case of structural discontinuities, result in misleading volcanological interpretations.

All equations, calculations, and most figures in this chapter are included in a Mathematica notebook, although some of the chapter figures are modified versions of those created in the notebook. Mathematica is one of several mathematics software packages capable of symbolic mathematics. Mathematica allows entry of equations in familiar typeset forms, rather than the more cryptic inline expressions typical of programming languages such as FORTRAN or C. In general, equations for surface displacements for each model are entered directly into Mathematica and derivatives, such as tilt and strain, are calculated directly. The Mathematica notebook that forms the basis of this chapter and a free reader to access the notebook are included in the DVD that accompanies this book.

8.2 THE ELASTIC HALF-SPACE: A FIRST APPROXIMATION OF THE EARTH

8.2.1 Properties of an isotropic linearly elastic solid

The mathematical source models we discuss represent the Earth's crust as an ideal semi-infinite elastic body, known as an elastic half-space. The half-space has one planar surface bounding a continuum that extends infinitely in all other directions. The half-space is materially homogeneous and mechanically isotropic (i.e., mechanical properties do not vary with direction), and it obeys Hooke's law, which specifies a linear relation between displacements (strains) at any point in the body and applied forces (stresses). A body with these characteristics is called an isotropic linearly elastic solid. In short-term laboratory tests, rocks behave like linear elastic solids for strains less than about 1% (10,000 ppm), particularly at low temperatures. Over long time periods or at high temperatures, a non-linear rheology is more appropriate for the crust. Elastic half-space models often neglect many characteristics of the real Earth, but they provide good approximations of deformation resulting from infinitesimal, short-term phenomena on the surface of, or within, the shallow crust.

More realistic Earth models have been developed to account for curvature, topography, gravity, vertical layering, lateral inhomogeneity, and time-varying material properties. We expect that continued advances in computing power and improvements in geodetic data quality and station density will eventually lead to widespread use of these more realistic models. We limit our discussion, however, to the simplest analytical elastic half-space models, partly because they are convenient but also because modeling buried sources is inherently ambiguous. We include a short discussion of topographic corrections. Even with these simple models, there are many trade-offs between model parameters. Our discussion emphasizes the assumptions inherent in the models and how the predicted deformation is related to model geometry, location, and pressure change or displacement. Our goal is to understand both the characteristics and limitations of these simple analytical models.

8.2.2 Elastic constants

The constitutive equations for an isotropic, linearly elastic solid need only two independent elastic constants to describe the relationship between stress and strain. Generally, volcanic source models use Poisson's ratio $\nu$ and the modulus of elasticity in shear $G$. The shear modulus, also called the rigidity modulus or the second Lamé constant, is often represented by $\mu$. We use $G$ to avoid confusion with the prefix used to represent 1 ppm (e.g., $\mu$strain). Okada (1985) and Okada (1992) use both Lamé's constants $\lambda$ and $G$, and:

$$\lambda = \frac{2G\nu}{1 - 2\nu}$$

What is the physical significance of $G$ and $\nu$? $G$ relates shear stress to strain providing a material's rigidity or 'stiffness' under shear and has units of