11 Numerical Solution of Ordinary Differential Equations

In this chapter we deal with the numerical solutions of the Cauchy problem for ordinary differential equations (henceforth abbreviated by ODEs). After a brief review of basic notions about ODEs, we introduce the most widely used techniques for the numerical approximation of scalar equations. The concepts of consistency, convergence, zero-stability and absolute stability will be addressed. Then, we extend our analysis to systems of ODEs, with emphasis on stiff problems.

11.1 The Cauchy Problem

The Cauchy problem (also known as the initial-value problem) consists of finding the solution of an ODE, in the scalar or vector case, given suitable initial conditions. In particular, in the scalar case, denoting by \( I \) an interval of \( \mathbb{R} \) containing the point \( t_0 \), the Cauchy problem associated with a first order ODE reads:

\[
\begin{align*}
\frac{dy}{dt}(t) &= f(t, y(t)), & t & \in I, \\
y(t_0) &= y_0,
\end{align*}
\]

(11.1)

where \( f(t, y) \) is a given real-valued function in the strip \( S = I \times (-\infty, +\infty) \), which is continuous with respect to both variables. Should \( f \) depend on \( t \) only through \( y \), the differential equation is called autonomous.

Most of our analysis will be concerned with one single differential equation (scalar case). The extension to the case of systems of first-order ODEs will be addressed in Section 11.9.

If \( f \) is continuous with respect to \( t \), then the solution to (11.1) satisfies

\[
y(t) - y_0 = \int_{t_0}^{t} f(\tau, y(\tau))d\tau.
\]

(11.2)
Conversely, if $y$ is defined by (11.2), then it is continuous in $I$ and $y(t_0) = y_0$. Moreover, since $y$ is a primitive of the continuous function $f(\cdot, y(\cdot))$, $y \in C^1(I)$ and satisfies the differential equation $y'(t) = f(t, y(t))$.

Thus, if $f$ is continuous the Cauchy problem (11.1) is equivalent to the integral equation (11.2). We shall see later on how to take advantage of this equivalence in the numerical methods.

Let us now recall two existence and uniqueness results for (11.1).

1. **Local existence and uniqueness.**
   Suppose that $f(t, y)$ is locally Lipschitz continuous at $(t_0, y_0)$ with respect to $y$, that is, there exist two neighborhoods, $J \subseteq I$ of width $r_J$, and $\Sigma$ of $y_0$ of width $r_\Sigma$, and a constant $L > 0$, such that
   \[
   |f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \quad \forall t \in J, \forall y_1, y_2 \in \Sigma. \quad (11.3)
   \]
   Then, the Cauchy problem (11.1) admits a unique solution in a neighborhood of $t_0$ with radius $r_0$ with $0 < r_0 < \min(r_J, r_\Sigma/M, 1/L)$, where $M$ is the maximum of $|f(t, y)|$ on $J \times \Sigma$. This solution is called the local solution.

   Notice that condition (11.3) is automatically satisfied if $f$ has continuous derivative with respect to $y$: indeed, in such a case it suffices to choose $L$ as the maximum of $|\partial f(t, y)/\partial y|$ in $J \times \Sigma$.

2. **Global existence and uniqueness.** The problem admits a unique global solution if one can take $J = I$ and $\Sigma = \mathbb{R}$ in (11.3), that is, if $f$ is uniformly Lipschitz continuous with respect to $y$.

In view of the stability analysis of the Cauchy problem, we consider the following problem

\[
\begin{aligned}
   z'(t) &= f(t, z(t)) + \delta(t), \quad t \in I, \\
   z(t_0) &= y_0 + \delta_0,
\end{aligned}
\quad (11.4)
\]

where $\delta_0 \in \mathbb{R}$ and $\delta$ is a continuous function on $I$. Problem (11.4) is derived from (11.1) by perturbing both the initial datum $y_0$ and the function $f$. Let us now characterize the sensitivity of the solution $z$ to those perturbations.

**Definition 11.1** ([Hah67] or [Ste71]). Let $I$ be a bounded set. The Cauchy problem (11.1) is stable in the sense of Liapunov (or stable) on $I$ if, for any perturbation $(\delta_0, \delta(t))$ satisfying

\[
   |\delta_0| < \varepsilon, \quad |\delta(t)| < \varepsilon \quad \forall t \in I,
\]

with $\varepsilon > 0$ sufficiently small to guarantee that the solution to the perturbed problem (11.4) does exist, then

\[
   \exists C > 0 \text{ such that } |y(t) - z(t)| < C\varepsilon, \quad \forall t \in I. \quad (11.5)
\]