Chapter XXIII. The Strictly Hyperbolic Cauchy Problem

Summary

The Cauchy problem for a constant coefficient differential operator $P$ with data on a non-characteristic plane is correctly posed for arbitrary lower order terms if and only if $P$ is strictly hyperbolic (Corollary 12.4.10 and Definition 12.4.11). If the plane is defined by $x_n = 0$ this means that the principal symbol $P_m(\xi', \xi_n)$ has $m$ distinct real zeros $\xi_n$ for every $\xi'$ $=(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} \setminus 0$. In Section 23.2 we shall prove that this condition also guarantees the correctness of the Cauchy problem in a very strong sense when the coefficients are variable. The converse will be discussed in Section 23.3 and in the notes.

The proofs in Section 23.2 depend on factorization of $P(x, D)$ into first order factors of the form $D_n - a(x, D')$ where the principal symbol of $a$ is one of the zeros of $P_m(x, \xi', \xi_n)$. We shall therefore study such first order operators in Section 23.1. The basic tool is the energy integral method. It could be applied directly to the operator $P(x, D)$ but it is more transparent and elementary in the first order case which will therefore be studied first.

In Section 23.3 we show that for operators of principal type the correctness of the Cauchy problem requires strict hyperbolicity away from the plane carrying Cauchy data. However, certain types of double roots may occur at that plane. This situation is studied at some length in Section 23.4 since it occurs for the important Tricomi equation. We shall also come across such operators in Section 24.6.

23.1. First Order Operators

In this section we shall study the Cauchy problem in $\mathbb{R}^{n+1}$

(23.1.1) $\frac{\partial u}{\partial t} + a(t, x, D)u = f$, \quad $0 < t < T$; \quad $u = \phi$ \quad when \quad $t = 0$.

We shall assume

(i) $a_t(x, \xi) = a(t, x, \xi)$ belongs to a bounded set in $S^1(\mathbb{R}^{n} \times \mathbb{R}^{n})$ when $0 \leq t \leq T$;
(ii) \( t \to a_t \) is continuous with values in \( \mathcal{S}(\mathbb{R}^{2n}) \) (or equivalently in \( C^\infty(\mathbb{R}^{2n}) \));

(iii) \( \Re a(t, x, \xi) \geq -C, \ 0 \leq t \leq T. \)

The assumption (iii) is natural for if \( a \) is constant and \( f = 0 \) then the solution of (23.1.1) is \( u = e^{-at} \phi \).

It follows from (i), (ii) that \( t \to a(t, x, D)u \) is continuous with values in \( \mathcal{S} \) if \( u \in \mathcal{S} \), for \( a(t, x, D)u \) is bounded in \( \mathcal{S} \) by Theorem 18.1.6 and continuity with values in \( C^\infty \) is obvious. In view of Theorem 18.1.13 it follows that \( a_t(x, D) \) is a strongly continuous function of \( t \) with values in the set of bounded operators from \( H^s(\mathbb{R}^n) \) to \( H^s(\mathbb{R}^n) \), for every \( s \).

The following basic estimate is proved by the energy integral method which we already encountered in the proof of Lemma 17.5.4.

**Lemma 23.1.1.** If \( s \in \mathbb{R} \) and if \( \lambda \in \mathbb{R} \) is larger than some number depending on \( s \), we have for every \( u \in C^1([0, T]; H^s(\mathbb{R}^n)) \cap C^0([0, T]; H^s(\mathbb{R}^n)) \) and \( p \in [1, \infty] \)

\[
(23.1.2) \quad \left( \frac{1}{2} \int_0^T \| e^{-\lambda t} u(t, \cdot) \|_{(s)}^p \lambda \, dt \right)^{1/p} \leq \| u(0, \cdot) \|_{(s)} + 2 \int_0^T e^{-\lambda t} \| \partial u / \partial t + a(t, x, D)u \|_{(s)} \, dt,
\]

with the obvious interpretation as a maximum when \( p = \infty \).

**Proof.** First assume that \( s = 0 \). By Theorem 18.1.14 it follows from conditions (i) and (iii) that for some constant \( c \)

\[
\Re (a_t(x, D) v, v) \geq -c(v, v), \quad v \in H^1(\mathbb{R}).
\]

Writing \( f = \partial u / \partial t + a(t, x, D)u \) and taking scalar products for fixed \( t \), we obtain

\[
2 \Re (f(t), u(t)) e^{-2\lambda t} = \frac{\partial}{\partial t} e^{-2\lambda t} \| u(t) \|^2 + 2 \Re ((a_t(x, D) + \lambda)u(t), u(t)) e^{-2\lambda t} \geq \frac{\partial}{\partial t} e^{-2\lambda t} \| u(t) \|^2
\]

provided that \( \lambda \geq c \). If we integrate from 0 to \( \tau \leq t \leq T \) it follows that

\[
M(t)^2 = \sup_{0 \leq \tau \leq t} e^{-2\lambda \tau} \| u(\tau) \|^2 \leq \| u(0) \|^2 + 2 M(t) \int_0^t e^{-\lambda \tau} \| f(\tau) \| \, d\tau.
\]

Hence

\[
\left( M(t) - \int_0^t e^{-\lambda \tau} \| f(\tau) \| \, d\tau \right)^2 \leq \left( \| u(0) \|^2 + \int_0^t e^{-\lambda \tau} \| f(\tau) \| \, d\tau \right)^2,
\]

which for \( \lambda = c \) implies

\[
e^{-ct} \| u(t) \| \leq \| u(0) \|^2 + 2 \int_0^t e^{-c\tau} \| f(\tau) \| \, d\tau.
\]

If we multiply by \( e^{(c-\lambda)t} \) it follows that

\[
e^{-\lambda t} \| u(t) \| \leq e^{(c-\lambda)t} \| u(0) \|^2 + 2 \int_0^t e^{-\lambda \tau} \| f(\tau) \| e^{(c-\lambda)(t-\tau)} \, d\tau.
\]