Chapter 9
A Posteriori Error Estimation

The error analysis for the Galerkin discretization exhibits the asymptotic convergence rates for the boundary element method which depend on the regularity of the underlying integral equation. These estimates are called a priori estimates because they hold for large classes of problems which are characterized by their regularity. They are important because they show the asymptotic quality of the Galerkin boundary method. However, for a concrete problem these estimates could be by far too pessimistic and do not allow answers to the following questions:

- Is the size of the error $u_\ell - u$ with respect to some norm or to some other measure below some given error tolerance $\varepsilon$?
- If the numerical solution $u_\ell \in S_\ell$ is not accurate enough, what is a good strategy to enrich the space $S_\ell$ in a problem-oriented way? Is the uniform refinement as described in Remark 4.1.8 a good strategy?

The a posteriori error estimation uses the computed numerical solution $u_\ell$ and the given data (such as the right-hand side or the integral operator) and computes non-negative indicators $(\eta_i)_{i=1}^n$ which have the property that the (weighted) sum is an upper bound for the true error. The quantities will be local in the sense that their computation involves integrals over small patches $\omega_i \subseteq \Gamma$ and their number $n$ depends linearly on the number of panels.

Furthermore, the size of these local quantities can be used directly to detect sub-regions on the surface $\Gamma$ where the error is large and which should then be locally refined.

The development of a posteriori error estimation for finite element discretizations of partial differential equations started with the pioneering papers [10, 11]. Since then the number of publications in this field has grown enormously and we refer to the monographs [2, 12, 14, 172, 232] for a thorough treatment of this topic and further references.

However, for boundary element methods, the nonlocal character of the integral operator and the nonlocal fractional Sobolev norms cause difficulties in the mathematical derivation of local error indicators and much fewer authors have investigated local a posteriori error estimates for integral equations [47, 48, 50–53, 89, 90, 92, 93, 189, 196, 209, 210, 240, 244].
In this chapter, we will develop and analyze a posteriori error estimators for boundary integral operators. We will follow the approach and the analysis as introduced in [89, 90, 92].

9.1 Preliminaries

In Chap. 4 we introduced the Galerkin boundary element method for the abstract variational problem: For given $F \in H'$, find $u \in H$ such that
\[ a(u, v) = F(v) \quad \forall v \in H. \]  
(9.1)

Let $A : H \to H'$ denote the operator associated with the sesquilinear form $a(\cdot, \cdot)$ (cf. Lemma 2.1.38). Throughout this chapter we will assume that the operator $A$ is either of negative order and maps into $H^s(\Gamma)$ for some positive $s$ or is of non-negative order. Note that we always require throughout this chapter that the range for the differentiability indices $s$ in $H^s(\Gamma)$ obeys condition (2.84) depending on the smoothness of $\Gamma$. A first assumption on the operator $A$ is stated next.

**Assumption 9.1.1.** $A : H^s(\Gamma) \to H^{-s}(\Gamma)$ is an isomorphism for some order $2s \in \mathbb{R}$, i.e., there exist constants $C_1, C_2 > 0$ such that
\[ \| A \|_{H^{-s}(\Gamma) \to H^s(\Gamma)} \leq C_1 \quad \text{and} \quad \| A^{-1} \|_{H^s(\Gamma) \to H^{-s}(\Gamma)} \leq C_2. \]

The boundary element space $S$ is composed by local polynomials which are lifted to the surface $\Gamma$ via local charts and put together either in a continuous or discontinuous way. The Galerkin discretization is given by seeking $u_S \in S$ such that
\[ a(u_S, v) = F(v) \quad \forall v \in S. \]  
(9.2)

The boundary element mesh is denoted by $\mathcal{G}$ consisting of surface panels $\tau$ (cf. Chap. 4).

Typically, the error $u - u_S$ will not be distributed uniformly over the surface $\Gamma$, and adaptive refinement aims at refining the mesh in regions where the error is larger than some threshold. In this chapter, we will introduce local a posteriori refinement indicators for the detection of such regions (and for the estimation of the total error). In this light, the goal of this chapter is to define computable quantities $\eta_i$ which will depend on the discrete solution $u_S$ such that the estimates
\[ C_{\text{eff}} \sum_{i=1}^n \eta_i^2 \leq \| u_S - u \|_{H^s(\Gamma)}^2 \leq C_{\text{rel}} \sum_{i=1}^n \eta_i^2 \]  
(9.3)
hold. The upper estimate is called “reliability” because it guarantees a prescribed given accuracy while the lower estimate is called “efficiency” because it implies that the qualitative behavior of the error is reflected by the error indicators and not