16 Canonical Approach to Asymmetric Continua

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16.1 Introduction

This chapter presents a canonical approach to the theory of asymmetric continua in terms of Lagrangians and Hamiltonians. The Lagrangian is expressed in terms of the spin structure and spin rates. Consequently, the Hamiltonian is described as a function of the spin structure and angular momenta (Majewski 2006a). Landau and Lifshitz (1958, 1960) gave a clear exposition of the Lagrangian and Hamiltonian formulations of classical mechanics. In their formulation for a system of material points, the Lagrangian is a function of generalized positions and generalized velocities of the material points. They viewed Hamiltonian as a function of the generalized positions and generalized momenta. Maugin (2003) formulated the canonical mechanics of a nonlinear elastic material. Kleinert (1988, 1989, 2008) elucidated an action approach to gravitational and electromagnetic fields. A Lagrangian formulation of an asymmetric elastic continuum was applied by Majewski (2006b) in the context of rotational seismic waves and accompanying them spin and twist solitons. Moreover, Majewski (2006c) used a Lagrangian approach in the framework of the gauge field theory of an elasto-plastic continuum with dislocations. In the gauge field theory the fundamental equations are obtained by variations of the gauge invariant Lagrangian.

The goal of the canonical approach in physics is to start from a single Hamilton’s Principle of Least Action and to find an extremum of an integral describing energy in order to derive differential equations of motion, called in mechanics Lagrange’s equations. In the variational calculus they are called Euler’s equations for the general mathematical problem of determining the extrema of an integral. The Lagrangian of a mechanical system represents the difference between its kinetic and potential energies and allows us to derive the equations of motion of the system. They relate coordinates, velocities and accelerations.
Mathematically speaking, the number of equations of motion is equal to the number of generalized coordinates, which are treated as unknown functions. Let us say that we have $n$ generalized coordinates. In such a case, the set of equations of motion comprises a set of $n$ second-order differential equations for $n$ unknown functions—generalized coordinates. The general solution has $2n$ arbitrary constants. In order to find these constants and to describe completely the motion of the system in question, we have to determine the initial conditions. By the initial condition we mean the initial numerical values of the coordinates and velocities. They define the state of the system at the initial time.

We should be aware that the construction of the Lagrangian is connected with a choice of variables. Consequently, the same variables will reappear in the derived Lagrange’s equations of motion. Classically, the generalized coordinates and velocities are used to construct the Lagrangian that describes the difference between the kinetic and potential energies of the mechanical system. We should emphasize that this is not the only possible choice of variables. Sometimes, particularly, when we need to know the total energy of the system, it is more convenient to describe the system in terms of the generalized coordinates and momenta. The best way to transfer from one set of variables to another set is to use the Legendre’s transform. In order to find the energy of a Lagrangian system, it is convenient to construct the so-called Hamiltonian. Using the Hamiltonian, we can obtain a new set of first-order differential equations of motion. These equations of motion are called Hamilton’s equations or canonical equations. In these equations, the unknown functions are the generalized coordinates and the generalized momenta of the system. They can be treated as evolution equations for the generalized coordinates and generalized momenta.

As far as we know, the canonical approach was applied to translational motions. Our goal, here, is to show how the canonical approach can be applied to rotational motions. Rotational motions can be realized in asymmetric media. Thus, our contribution consists in the application of a different set of variables: spin structure and spin rates to create the Lagrangian, and the spin structure and angular momenta to form the Hamiltonian.

### 16.2 Hamilton’s Principle

The fundamental variational principle of mechanics is Hamilton’s Principle that can be viewed as a Principle of Least Action on intuitive grounds. Hamilton’s Principle states that from all possible paths of motion