6 Field Invariant Representation: Dirac Tensors

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6.1 Introduction

Using the Dirac matrices (Dirac 1982, Good 1955) we can give invariant presentation of any symmetric and antisymmetric tensor forms. Splitting a symmetric tensor into the axial and deviatoric parts we come, with the help of appropriate Dirac matrices, to their invariant representation in any system. Here we present the diagonal and off-diagonal tensor forms of the deviatoric tensors.

6.2 Axial and Deviatoric Parts of any Symmetric Tensor

In Chapter 1 we have discussed the basic motions in an asymmetric continuum: the physical fields are represented by tensors. Any antisymmetric tensor preserves its general form; however, the symmetric tensors may have different forms in different systems: diagonal form, off-diagonal or mixed one. Therefore, it is useful to split a symmetric tensor into its axial, $E^A_{ik}$, and deviatoric, $E^D_{ik}$, parts:

$$E_{ik} = E^A_{ik} + E^D_{ik} = \frac{1}{3} E_{ss} \delta_{ik} + \left( E_{ik} - \frac{1}{3} E_{ss} \delta_{ik} \right). \quad (6.1)$$

Of course, a tensor with vanishing trace is equal to its deviatoric part:

$$\omega_{(ik)} = \omega^D_{(ik)}, \quad \text{at} \quad \omega_{(ss)} = 0. \quad (6.2)$$

In a specially chosen coordinate system we can describe a deviatoric tensor (6 values in general) in its diagonal or off-diagonal forms (3 values only):

$$\omega^D_{(ik)} = \begin{bmatrix} \omega^D_{(11)} & 0 & 0 \\ 0 & \omega^D_{(22)} & 0 \\ 0 & 0 & \omega^D_{(33)} \end{bmatrix}, \text{ or } \omega^D_{(ik)} = \begin{bmatrix} 0 & \omega_{(3)} & -\omega_{(2)} \\ \omega_{(3)} & 0 & \omega_{(1)} \\ -\omega_{(2)} & \omega_{(1)} & 0 \end{bmatrix} \quad (6.3)$$
where we have put $\omega_{(12)}^D = \omega_{(3)}$, $\omega_{(13)}^D = -\omega_{(2)}$, and $\omega_{(23)}^D = \omega_{(1)}$.

When comparing the observational/recording data related to a symmetric tensor with the theoretical results presented in such a special system, we shall be aware that we must transform the measured tensor values to the appropriate system (diagonal or off-diagonal) for any time moment. However, for a theoretical consideration, there exists a method to preserve such forms invariant with the help of Dirac tensors, maintaining these forms (diagonal or off-diagonal) in any 4D system. We will consider the off-diagonal case.

### 6.3 Dirac Tensors

We introduce the following system of the Dirac $\varepsilon^\alpha$ tensors:

$$
\varepsilon^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \varepsilon^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
$$

$$
\varepsilon^3 = i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \varepsilon^4 = i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
$$

For the system \(\{x_s, i\sigma t\}\), $\eta^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, these tensors fulfil the following condition:

$$
\frac{1}{2} (\varepsilon^\alpha \varepsilon^\beta + \varepsilon^\beta \varepsilon^\alpha) = \eta^{\alpha\beta}.
$$

Some other Dirac tensors can be obtained as their products, e.g.:

$$
\varepsilon^1 \varepsilon^3 = i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \varepsilon^2 \varepsilon^3 = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.
$$