A Comparative Study of Linear and Semidefinite Branch-and-Cut Methods for Solving the Minimum Graph Bisection Problem

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Abstract. Semidefinite relaxations are known to deliver good approximations for combinatorial optimization problems like graph bisection. Using the spectral bundle method it is possible to exploit structural properties of the underlying problem and to apply, even to sparse large scale instances, cutting plane methods, probably the most successful technique in linear programming. We set up a common branch-and-cut framework for linear and semidefinite relaxations of the *minimum graph bisection problem*. It incorporates separation algorithms for valid inequalities presented in the recent study [2] of the facial structure of the associated polytope. Extensive numerical experiments show that the semidefinite branch-and-cut approach outperforms the classical simplex approach on a clear majority of the sparse large scale test instances. On instances from compiler design the simplex approach is faster.

Keywords: Branch and cut algorithms, cutting plane algorithms, polyhedral combinatorics, semidefinite programs.

1 Introduction

Let $G = (V, E)$ be an undirected graph with $V = \{1, \dots, n\}$ and $E \subseteq \{\{i, j\} : i, j \in V, i < j\}$. For given vertex weights $f_v \in \mathbb{N}_\cup\{0\}$, $v \in V$, and edge costs $w_{\{i,j\}} \in \mathbb{R}$, $\{i, j\} \in E$, a partition of the vertex set V into two disjoint clusters S and $V \setminus S$ with sizes $f(S) \le F$ and $f(V \setminus S) \le F$, where $F \in \mathbb{N} \cap \left[\frac{1}{2} f(V), f(V)\right]$, is called a *bisection*. Finding a bisection such that the total cost of edges in the cut $\delta(S) := \{\{i, j\} \in E : i \in S \wedge j \in V \setminus S\}$ is minimal is the *minimum bisection problem* (MB). The problem is known to be NP-hard [9]. The polytope associated with MB,

$$P_{\mathrm{B}} := \operatorname{conv}\left\{ y \in \mathbb{R}^{|E|} : y = \chi^{\delta(S)},\, S \subseteq V,\, f(S) \le F,\, f(V \setminus S) \le F \right\},$$

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where $\chi^{\delta(S)}$ is the incidence vector of the cut $\delta(S)$ with respect to the edge set E, is called the *bisection cut polytope*. MB as well as P_{B} are related to other problems and polytopes already known in the literature. Obviously, the bisection cut polytope is contained in the *cut polytope* [3,6]

$$P_{\mathrm{C}} := \operatorname{conv}\left\{ y \in \mathbb{R}^{|E|} : y = \chi^{\delta(S)}, S \subseteq V \right\}.$$

If $F = f(V)$ then MB is equivalent to the maximum cut problem (using the negative cost function) and $P_{\mathrm{B}} = P_{\mathrm{C}}$. For $F = \lceil \frac{1}{2} f(V) \rceil$ MB is equivalent to the *equipartition problem* [5] and the bisection cut polytope equals the *equipartition polytope* [4,15]. Furthermore, MB is a special case of the minimum node capacitated graph partitioning problem (MNCGP) [8], where more than two clusters are available for the partition of the node set and each cluster has a common limited capacity. The objective in MNCGP is the same as in MB, i.e., to minimize the total cost of edges having endpoints in distinct clusters. Finally, we mention the *knapsack polytope* [21]

$$P_{\mathrm{K}} := \operatorname{conv}\left\{ x \in \{0,1\}^{|V|} : \sum_{v \in V} f_v x_v \leq F \right\},$$

which plays a fundamental role in valid inequalities for P_{B}. Graph partitioning problems in general have numerous applications, for instance in numerics [10], VLSI-design [18], compiler-design [16] and frequency assignment [7]. A large variety of valid inequalities for the polytope associated with MNCGP is known [3,4,8,15] and, since MB is a special case of MNCGP, all those inequalities are also valid for P_{B}. A recent successful study of a combined semidefinite polyhedral branch-and-cut approach for max-cut is [20], it is designed for rather dense graphs with up to 400 nodes. In contrast, our semidefinite branch-and-cut approach is applicable to sparse graphs with up to 2000 nodes. In addition, we present a direct comparison with an LP approach within the same branch-and-cut environment where both approaches use the same separation routines.

In [2] we give a detailed analysis of P_{B} including several classes of new and facet-defining inequalities. We summarize these results and those from the literature in Sect. 2. We use these inequalities to derive and strengthen two relaxations for MB. One is based on an integer programming, the second on a semidefinite programming formulation. We develop in Sect. 3 both an LP-based branch-and-cut algorithm and an SDP based branch-and-cut algorithm using the same framework SCIP [1]. In Sect. 4 we give a comprehensive computational comparison of both approaches on various test instances with some surprising outcomes.

2 Valid Inequalities for P_{B}

A large variety of valid inequalities for the cut polytope, the equipartition polytope, and the polytope associated with MNCGP is known: cycle inequalities [3] of the cut polytope; tree, star, and cycle inequalities [4] as well as suspended