7 Risk Measurement with Spectral Capital Allocation

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Spectral risk measures provide the framework to formulate the risk aversion of a firm specifically for each quantile of the loss distribution of a portfolio. More precisely the risk aversion is codified in a weight function, weighting each quantile. Since the basic coherent building blocks of spectral risk measures are expected shortfall measures, the most intuitive approach comes from combinations of those. For investment decisions the marginal risk or the capital allocation is the sensible approach. Since spectral risk measures are coherent there exists also a sensible capital allocation based on the notion of derivatives or more in the light of the coherency approach as an expectation under a generalized maximal scenario.

7.1 Introduction

Portfolio modeling has two main objectives: the quantification of portfolio risk, which is usually expressed as the economic capital of the portfolio, and its allocation to subportfolios and individual transactions. The standard approach in credit portfolio modeling is to define the economic capital in terms of a quantile of the portfolio loss distribution

\[ q_\alpha(L) = F_L^{-1}(\alpha). \]

The capital charge of an individual transaction is traditionally based on a covariance technique and called volatility contribution. We refer to Bluhm et al. (2002) and Crouhy et al. (2000) for a survey on credit portfolio modeling and capital allocation.

Since the work by Artzner et al (1997) coherent risk measures are discussed intensively in finance and risk management. More recent is the question of a more coherent capital allocation. Especially the use of expected shortfall allocation as an allocation rule is recommend in Overbeck (2000). Denault
Expected shortfall measures

\[ ES_\alpha(L) = \frac{1}{1 - \alpha} \int_0^1 q_u(L) du \]

are the building blocks of more general coherent risk measures, the spectral risk measure \( \rho \). These are convex mixtures of expected shortfall measures. They can be represented by their spectral measure \( \mu \) through

\[ \rho = \rho_\mu = \int_0^1 ES_\alpha(1 - \alpha) \mu(da) \quad (7.1) \]

or as a weighted sum of quantiles with \( w(\alpha) = \mu([0, \alpha]) \),

\[ \rho = \rho_\mu = \rho_w = \int_0^1 q_\alpha(\cdot) w(\alpha) d\alpha. \quad (7.2) \]

In this paper we apply the allocation rules associated with a spectral risk measure to a credit portfolio and point out, which consequences to risk management the choice of the weight function \( w \), the spectral measure \( \mu \) or the measure

\[ \tilde{\mu} \overset{\text{def}}{=} (1 - \alpha) \mu(d\alpha), \]

which we call mixing measure and thought to be the most easily one to calibrate and implement. The theoretical basis of the approach can be found in the basic papers Kalkbrener (2002), Kalkbrener et al (2004) and the explicit application to spectral capital allocation is provided by Overbeck (2005). We will first present the theoretical foundation of the proposed risk and allocation measures and then discuss general impact of the choice of the weight or mixing function and finally exhibits the differences on a concrete credit portfolio example.

### 7.2 Review of Coherent Risk Measures and Allocation

#### 7.2.1 Coherent Risk Measures

It is well-known that the following four conditions define a coherent risk measure, Artzner et al (1997, 1999), Delbaen (2000).