1 Introduction to Chaos Control: An Interdisciplinary Problem

1.1 Chaos Control Is Suppression or Synchronization

The foundational for chaos control problem is scientific as well as technological. In regard science, on the one hand, chaos control has two important contributions: (i) The controlled chaotic systems has allowed to understand that structured disorder and its entropy/information relationship extend the concept of determinism [1], [2] and (ii) departing from chaotification (inverse action of the chaos suppression) some questions have been opened on phenomena of the feedback dynamical systems [3]. Moreover, the chaos control impacts biomedical, life and engineering sciences; for example, it can be extended to control pathological rhythm in heart [4]. Now, regarding technological applications, the controlled chaotic systems are important because of a desired frequency response can be induced. Nowadays, the scientific community has identified two problems in chaos control: suppression and synchronization. Among others, we can mention studies in physical devices (e.g., telescopes or lasers), biology/ecology (e.g., population dynamics or biodynamics) or biomedical systems (e.g., heart rhythm or brain activity). Thus, for instance, controlled current-modulation can be entered as excitation from a nonlinear circuit into semiconductors lasers by feeding back the laser frequency response (see Figure 1 in [5]). Henceforth, scientific community has taken possession of the challenge of exploring control techniques such that (i) a family of driving force can command classes of chaotic systems [6], (ii) the synthesis of mathematical expressions for the control force accounts the frequency response [7], and (iii) energy requirements by the control force are accounted (for example to avoid saturation or deterioration in control devices) [8]. In addition, the mathematical models of the driving force is desired to be simple and easy to implement experimentally. A simple form is the linear models of driving forces; which can be expressed in the frequency (Laplace) or time domain and they have been already used to suppress chaotic behavior [7], [9].

In grosso, the chaos suppression problem can be defined as the stabilization of unstable periodic orbits (UPO's) of a chaotic attractor in equilibrium points or periodic orbits with period $n$ embedded into the chaotic attractor [10]. Since the
seminal paper by Ott, Grebogi and Yorke [11] was published, several control schemes have been proposed to suppress chaos. Continuous- and discrete-time approaches can be found in open literature (see, for example, [12] and [13]). Some feedback controllers have been designed from robustness against noisy environment [14]. Others have been proposed as robust approaches for state feedback control [15] and few schemes have been designed in frequency domain. In this sense, integral actions have shown capability to stabilize chaotic systems in equilibrium points and periodic orbits [7], [9]. Nevertheless, the control cost is often omitted in reports of chaos suppression. Thus, the following question rises: can we design a feedback control (driving force) with robustness and optimality issues for chaos suppression? The problem is not an easy task if we consider that: (i) Controversy on robust, optimal and fragility issues is open in control theory, (ii) Chaotic systems are, by nature, highly sensitive to initial conditions and parametric variations and (iii) The chaotic systems are nonlinear with continuous spectrum in frequency which can complicate the synthesis of the frequency domain driving forces.

Synchronization of chaotic systems is an interesting topic that, since early 90’s, has caught the attention of the nonlinear science community. Two research directions have been already conformed in synchronizing chaos: (i) analysis and (ii) synthesis. Analysis problem comprises (a) the classification of synchronization phenomena [16], [17]; (b) the comprehension of the synchronization properties as, for instance, robustness [18] or geometry [19], [20]; and (c) the construction of a general framework for unifying chaotic synchronization [17], [21]. On the other hand, synthesis of synchronization systems concerns the problem of finding the control force such that two chaotic systems share time evolution in some sense. Both analysis and synthesis directions are active research areas and one of the current challenges is to achieve and explain synchronization of chaotic system with different model. In fact, the study of the chaotic synchronization with different models makes sense in several systems (see references within [22], [23], [24] and [25]). Among others, we can account those with different fractal dimension [22], neural levels [23],[24], message transmission [25] or respiratory/circulatory coupling [24].

In regard to analysis in strictly different systems, the studies have been focused on the existence of synchronization manifolds for coupled systems and such manifolds strongly depend on measures from Lyapunov exponents [19], [20]. Synchronization of different models has been analysed in nonidentical space-extended systems (for the case of parameter mismatching)[26] and structurally nonequivalent system including delay [22]. In [19] chaotic synchronization has been also analysed from invariant manifolds in terms of the existence of a diffeomorphism between the attractor of the coupled systems; which is closely related to generalized synchronization (GS). Josic [19] had included synchronization of different systems, and illustrative examples show the existence of synchronization manifolds; e.g. between Rössler and Lorenz. This analysis departs from rigorous definitions, and is deep for the complete synchronization (i.e., the synchronization of all master states with all corresponding states of the slave system, [16]). Unfortunately, such a formalism for other synchronization phenomena (as, for example, the partial-state synchronization [16]) is still obscure.

Concerning the synthesis, by the end of 90’s [27], some efforts have been done to synchronize chaotic systems with different model. The underlying idea is to find a synchronization force such that the existence of a synchronization manifold can be