1. Introduction.

In this chapter we present a circle of ideas introduced by Witten leading to a conjecture, bearing his name, regarding the intersection numbers of tautological classes on $\overline{M}_{g,n}$. As conjectured by Witten and first proved by Kontsevich, the generating series $F$ of these numbers satisfies differential equations

\begin{equation}
L_n(e^F) = 0, \quad n \geq -1. \tag{1.1}
\end{equation}

To be more specific, the intersection numbers at the center of Witten's conjecture are the numbers $<\tau_{d_1} \cdots \tau_{d_n}>$ defined by

\begin{equation}
<\tau_{d_1} \cdots \tau_{d_n}> = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \tag{1.2}
\end{equation}

where one sets $<\tau_{d_1} \cdots \tau_{d_n}> = 0$ if $d_1 + \cdots + d_n \neq 3g - 3 + n$. As we observed in Section 4 of Chapter XVII, it is remarkable that the knowledge of the numbers (1.2) for all $g$ and $n$ implies the knowledge of all the intersection numbers among all the tautological classes for all $g$ and $n$. We recall that, by definition, the tautological classes are the boundary classes, $\delta_{irr}$ and $\delta_{a,A}$, the Mumford classes $\kappa_{\nu} \in H^{2\nu}(\overline{M}_{g,n}, \mathbb{Q})$, and the point bundle classes $\psi_i \in H^2(\overline{M}_{g,n}, \mathbb{Q})$, $i = 1, \ldots, n$. The generating series for intersection numbers (1.2) is the formal power series

\begin{equation}
F(t_0, t_1, \ldots) = \sum_{d_1 \geq 0, \ldots, d_n \geq 0} \frac{1}{n!} <\tau_{d_1} \cdots \tau_{d_n} > t_1^{d_1} \cdots t_n^{d_n}. \tag{1.3}
\end{equation}

The differential operators $L_n$ in (1.1) are defined by

\begin{align*}
L_n &= -(2n + 3)!! \frac{\partial}{\partial t_{n+1}} + \sum_{i=0}^{\infty} \frac{(2n + 2i + 1)!!}{(2i - 1)!!} t_i \frac{\partial}{\partial t_{i+n}} \\
&\quad + \frac{1}{2} \sum_{r+s+1=n} (2r + 1)!!(2s + 1)!! \frac{\partial^2}{\partial r \partial s}. 
\end{align*}
In Section 2 of this chapter we show how equations (1.1) allow one to recursively compute all the intersection numbers (1.2), starting from the first one,

\[
<\tau_0\tau_0\tau_0> = \int_{\overline{M}_{0,3}} 1 = 1.
\]

In Section 3 we describe the connection between equations (1.1) and the KdV hierarchy. This is very important from a conceptual point of view in that it can be used as one of the tools to prove Witten’s conjecture (1.1), and also from a more practical point of view if one is interested in the actual computation of the intersection numbers (1.2).

An important remark is that the operators \(L_n\) satisfy the Virasoro-type commutation relations

\[
[L_m, L_n] = (m - n) L_{m+n} \quad \text{for } n, m \geq -1.
\]

For this reason, equations (1.1) are often referred to as Virasoro equations. The first two equations \(L_{-1}(e^F) = 0\) and \(L_0(e^F) = 0\) are called, respectively, the string equation and the dilaton equation. Looking at what these equations mean at the level of intersection numbers, one finds exactly the relations we already proved in Proposition (4.9) of Chapter XVII. In view of the commutation relations (1.4), in order to prove Witten’s conjecture (1.1), it suffices to prove the single equation \(L_2(e^F) = 0\). We will verify this relation in Section 6. To do so, we will have to transform the generating series \(F\) to apply effectively the operator \(L_2\) to \(e^F\).

In Section 4, we come to Kontsevich’s idea of using the cellular decomposition of \(M_{g,n}\) we described in Chapter XVIII. The first step is to give a combinatorial expression \(\omega_i\) for the cohomology class \(\psi_i\) in terms of this cellular decomposition. This is what we did in Section 7 of Chapter XIX (cf. formula (7.5) in that chapter). Then

\[
<\tau_{d_1} \cdots \tau_{d_n}> = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{M_{g,n}^{\text{comb}(r)}} \omega_1^{d_1} \cdots \omega_n^{d_n},
\]

and one can hope that the right-hand side of this equality can be expressed solely in terms of ribbon graphs. This is indeed possible, and after nontrivial combinatorial computations, also involving Theorem (8.8) in Chapter XIX, one can prove the following remarkable statement. Let \(\Lambda_1, \ldots, \Lambda_N\) be variables in \(\mathbb{R}^N_+\). Set \(\Lambda = \text{diag} (\Lambda_1, \ldots, \Lambda_N)\) and

\[
t_i(\Lambda) = -(2i - 1)!! \text{Tr} \left( \Lambda^{-(2i+1)} \right) = -(2i - 1)!! \left( \Lambda_1^{-(2i+1)} + \cdots + \Lambda_N^{-(2i+1)} \right).
\]

Look at the generating series (1.3) in the new variables \(\Lambda_1, \ldots, \Lambda_N\). Then

\[
F(t_0(\Lambda), t_1(\Lambda), \ldots) = \sum_{G \in \mathcal{G}^{1,c}} \frac{(\sqrt{-1}/2)^{|X_0(G)|}}{\text{Aut}(G)} \prod_{e \in X_1(G)} \frac{2}{\Lambda_e + \Lambda_{e'}},
\]