Chapter XIII.

Line bundles on moduli

1. Introduction.

This chapter is devoted to the study of quasi-coherent sheaves, and in particular invertible ones, on the moduli stacks of curves. There are several equivalent notions of quasi-coherent sheaf on a stack $\mathcal{M}$. According to the basic one, such an animal consists of the datum, for any scheme $S$ and any object $\xi$ in $\mathcal{M}(S)$, of a quasi-coherent sheaf $\mathcal{F}_\xi$ on $S$, plus suitable compatibility conditions between the various $\mathcal{F}_\xi$. When $\mathcal{M}$ is a scheme $X$, this is equivalent to the usual definition of quasi-coherent sheaf on $X$. For an object $\xi \in \mathcal{M}(S)$, that is, for a morphism $\xi : S \to X$, the sheaf $\mathcal{F}_\xi$ is just the pullback $\xi^* \mathcal{F}$ of $\mathcal{F} = \mathcal{F}_{\text{id}_X}$.

When $\mathcal{M}$ is a Deligne–Mumford stack, an alternative approach comes from the following observation. Giving a quasi-coherent sheaf on a scheme $Z$ is equivalent to giving its restriction to every open subset of $Z$, plus compatibility conditions, that is, a description of how the various restrictions fit together. The analogues of open sets for a Deligne–Mumford stack $\mathcal{M}$ are schemes étale over $\mathcal{M}$, and one might define a quasi-coherent sheaf on $\mathcal{M}$ to be a quasi-coherent sheaf on each one of these, plus suitable compatibility conditions. A variant of this definition would be to look at an atlas $X \to \mathcal{M}$ and to define a quasi-coherent sheaf on $\mathcal{M}$ to be a quasi-coherent sheaf on $X$ endowed with suitable “descent data.” In Section 2 we show that on a Deligne–Mumford stack the above three notions of quasi-coherent sheaf are equivalent. The proof is a simple application of descent.

For a Deligne–Mumford stack, there is a well-defined notion of Picard group, and in Section 2 we begin to study the Picard groups of the moduli stacks $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves. We first introduce some of the natural or, as one says, tautological line bundles and vector bundles on $\overline{\mathcal{M}}_{g,n}$. The first one is the Hodge vector bundle $\mathcal{E}$ whose “fiber” at a curve $C$ is the vector space of abelian differentials $H^0(C, \omega_C)$. The Hodge line bundle is the top exterior power of $\mathcal{E}$; its class in the Picard group of $\overline{\mathcal{M}}_{g,n}$ is commonly designated by $\lambda$. Further natural line bundles are the point bundles $\mathcal{L}_i$, $i = 1, \ldots, n$. The fiber of $\mathcal{L}_i$ at a stable $n$-pointed curve $(C; x_1, \ldots, x_n)$ is the cotangent space to $C$ at $x_i$, and the class of $\mathcal{L}_i$ in the Picard group of $\overline{\mathcal{M}}_{g,n}$ is denoted by $\psi_i$. Finally, one may attach a line bundle $\mathcal{O}(\mathcal{D})$ to any Cartier divisor $\mathcal{D}$ on $\overline{\mathcal{M}}_{g,n}$; doing this for the boundary or for its irreducible components produces other natural line bundles.
bundles on moduli; the class of the line bundle attached to the boundary is usually denoted by $\delta$. In general, none of these line bundles descends to a line bundle on the coarse moduli space $\overline{M}_{g,n}$, the reason being that the automorphism group of $(C; x_1, \ldots, x_n)$, when nontrivial, acts nontrivially on the fiber of the relevant line bundle at $(C; x_1, \ldots, x_n)$. The section closes with an alternative description of quasi-coherent sheaves on the quotient stack $[H/G]$ of a scheme $H$ modulo the action of an algebraic group $G$. We show that giving a quasi-coherent sheaf on $[H/G]$ is the same as giving a $G$-equivariant quasi-coherent sheaf on $H$. Roughly speaking, such an object consists of a quasi-coherent sheaf $\mathcal{F}$ on $H$ together with a lifting to $\mathcal{F}$ of the action of $G$ on $H$. In particular, this shows that $\text{Pic}([H/G])$ coincides with the group $\text{Pic}(H,G)$ which parameterizes isomorphism classes of $G$-equivariant line bundles on $H$.

In Section 3 we discuss the tangent and cotangent bundles to the moduli stack $\overline{M}_{g,P}$, we study the normal sheaf $N_\xi \Gamma$ to a clutching morphism $\xi \Gamma: \overline{M}_\Gamma \to \overline{M}_{g,P}$, and we compute the “excess intersection bundles” for the intersections of two boundary strata.

One way of producing line bundles on moduli spaces is via the determinant construction. This is what we discuss in Section 4. In general, given a (connected) scheme $S$ and a vector bundle $F$, the determinant of $F$ is a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $\det F = (\wedge^{\text{max}} F, \text{rank } F)$, where the rank is to be taken modulo 2. The introduction of a grading is essentially forced when comparing $\wedge^{\text{max}} (F \oplus G)$ with $\wedge^{\text{max}} (G \oplus F)$. The notion of determinant naturally extends to a finite complex $F^\bullet$ of vector bundles over $S$. The determinant of $F^\bullet$ is defined to be

$$\det(F^\bullet) = \cdots \otimes (\det F^q)^{(-1)^q} \otimes (\det F^{q-1})(-1)^{q-1} \otimes \cdots.$$ 

Now if $\pi: X \to S$ is a family of nodal curves, and $\mathcal{F}$ a coherent sheaf on $X$, one may attempt to define the determinant of the cohomology of $\mathcal{F}$ as

$$d_\pi(\mathcal{F}) = \det(R\pi_* \mathcal{F}).$$

This means that, when $R\pi_* \mathcal{F}$ is quasi-isomorphic to a finite complex $C^\bullet$ of vector bundles, one sets $d_\pi(\mathcal{F}) = \det(C^\bullet)$. The properties of determinants ensure that this is independent of the particular complex $C^\bullet$ chosen. More generally, suppose that $S$ can be covered with open sets $U$ such that the restriction of $R\pi_* \mathcal{F}$ to $U$ is quasi-isomorphic to a finite complex $C^\bullet_U$ of vector bundles. Then the line bundles $\det(C^\bullet_U)$ patch together on the overlaps of the open sets $U$, yielding a line bundle on $S$, which we again call determinant of the cohomology of $\mathcal{F}$ and denote by $d_\pi(\mathcal{F})$. In practice, in defining the determinant of the cohomology we

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