2 Description of state of strain

In this section geometrical description of deformation of continuum, which is equally suitable for any type of solid, is considered. Basic equations concerning strain tensors and their invariants, as well as changes in geometrical objects undergoing deformation and the expressions for “tracking” loads are discussed. Simplifications which lead to the small deformations theory are also considered in this section.

2.1 Strain tensor and its invariants

Deformation of a solid is described using either Lagrangian or Eulerian co-ordinates. Location of material points of a solid after deformation may be identified using Cartesian co-ordinate system with unit vectors \( y^k \equiv y_k \). The relation between co-ordinates \( x_k \equiv x^k \) and \( X_k \equiv X^k \) of the same material point respectively before and after deformation may be expressed as

\[
\begin{align*}
x_k &= x_k(X_1, X_2, X_3, \tau) \\
X_n &= X_n(x_1, x_2, x_3, \tau)
\end{align*}
\] (1.117) (1.118)

Single-valued transformations (1.118) determine a deformation of the solid. Co-ordinates \( x_k \), known as Lagrangian, are associated with every point of the solid and do not change during deformation. Co-ordinate lines \( x_n \) in the undeformed state are being co-ordinate lines of Cartesian co-ordinate system, after deformation they transform into co-ordinate lines of curvilinear co-ordinate system. Co-ordinates \( X_n \), known as Eulerian, are associated with every point of the space and characterise deformation of material points moving through particular spatial point. They change during deformation. Co-ordinate lines \( X_n \) after deformation are being co-ordinate lines of Cartesian co-ordinate system, while before deformation they were co-ordinate lines of curvilinear co-ordinate system.

Geometrical description of deformation using Eulerian co-ordinates was considered, for example, in [22], where it was shown that statements of solid mechanics problems in Eulerian co-ordinates can be reduced to problem statements in Lagrangian co-ordinates, if appropriate relation between the invariants of Green and Almansi strain tensors was established. Thus, there is no loss in generality in considering problems in Lagrangian co-ordinates only. Lagrangian co-ordinates seem to be more natural for stating stability problems, because we usually know
disposition and shape of a solid in its natural (undeformed) state rather than after stability loss.

Taking into account the above considerations, let us associate with each point of a solid sets of three parameters, \(x^a\) or \(\theta^a\), assuming that a single-value correspondence between the points of a solid and sets of three parameters, \(x^a\) or \(\theta^a\), is retained at any moment of time \(\tau\). In the undeformed (natural) state these sets coincide with co-ordinates \((x^1, x^2, x^3)\) and \((\theta^1, \theta^2, \theta^3)\), considered in the previous section. We may regard co-ordinate systems \((x^1, x^2, x^3)\) and \((\theta^1, \theta^2, \theta^3)\) as if “frozen” into a solid.

Let the point of a solid, position of which in the undeformed (natural) state is given by the vector \(r\), undergo displacement \(u\). Its position in the deformed state is then identified by the vector \(r^*\) (henceforth an asterisk is put beside the quantities if they are considered in the deformed state)

\[
r^*(\theta^a, \tau) = r(\theta^a) + u(\theta^a, \tau)
\]

Covariant base vectors in the deformed state, given (1.119), (1.96) and (1.15), are defined as

\[
g^*_n = \frac{\partial r^*}{\partial \theta^a} = g_n + g_m \nabla_n u^m = g_n + g^m \nabla_n u_m = g_m \left( g^m_n + \nabla_n u^m \right)
\]

Similarly to (1.28), covariant components of the metric tensor in the deformed state, given (1.120), are defined as

\[
g^*_{nm} = g^*_n \cdot g^*_m = g_r \left( g^k_n + \nabla_n u^k \right) \cdot g_l \left( g^l_m + \nabla_m u^l \right) = g_{nm} + \nabla_m u_n + \nabla_n u_m + \nabla_n u^p \nabla_m u^q
\]

In accordance with (1.29), we introduce contravariant components of the metric tensor in the deformed state as

\[
g^*_{nk} g^*_{km} = \delta^m_n
\]

Similarly to (1.30), we may express contravariant components of the metric tensor in the deformed state as

\[
g^*_{nm} = \frac{1}{g^*} \frac{\partial g^*}{\partial g^*_{nm}}, \quad g^* = \det \|g^*_{pq}\|, \quad \frac{1}{g^*} = \det \|g^*_{pq}\|
\]

In view of (1.123), contravariant base vectors in the deformed state are introduced as

\[
g^* = g^*_{mn} g^*_n
\]