On Asymptotically Optimal Algorithm for One Modification of Planar 3-dimensional Assignment Problem

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Summary. In the paper the m-layer planar 3-dimensional assignment problem is considered. It is an NP-hard modification of well-known planar 3-dimensional assignment problem. Approximation algorithm, proposed by Gimadi and Korkishko is analysed. Its asymptotical optimality for a special class of random instances is proved.

1 Introduction

Let $C = (c_{ijk})$ be a $n \times n \times n$ 3-dimensional matrix with nonnegative elements. The problem of finding n transpositions π^1,...,π^n with constraints $\pi^i(k) \neq \pi^j(k)$ for all $1 \leq i < j \leq n$ which minimize (or maximize) the sum $\sum_{i=1}^{n} \sum_{k=1}^{n} \pi^i(k)$ is a well-known planar 3-dimensional assignment problem (3-PAP).

If the first size of input matrix and the number of required transpositions is decreased to m, where $1 < m < n$, the problem is called m-layer planar 3-dimensional assignment problem (m-3-PAP). Though this modification is a simplification of the problem (one can interpret m-3-PAP as 3-PAP with $c_{ijk} = 0$ for $i > m$) it remains NP-hard problem (see f.e. [1]).

For 3-PAP an approximation algorithm was constructed by Kravtsov and Krachkovsky ([5]), but the proof of its asymptotical optimality for the problem with random instances failed (see [8]). For m-3-PAP an approximation algorithm was proposed in [3] and its asymptotical optimality was proved for the case when $m \sim \ln n$ and c_{ijk} has uniform distribution. Below this algorithm is exposed to more thorough analysis and its asymptotical optimality is proved for the case when $m \sim n^\theta$ ($\theta = const$) and c_{ijk} has minorized type distribution function. Also performance estimates of the algorithm are improved.

2 Approximation Algorithm A for m-3-PAP

Below we consider m-3-PAP formulated as above.

2.1 Algorithm A Description

Stage i, i = 1,...,m.

Consecutively for $j = 1,...,n - 2(i - 1)$ do:

1. Find $k_j = argmin_{k=1,...,n} c_{ijk}$.
2. Set $\pi^i(j) = k_j$.
3. Forbid (set to infinity) elements $c_{ij'k_j}$ for $j' = j + 1,...,n$ and $c_{i'jk_j}$ for $i' = i + 1,...,m$.

Use Hopkroft-Karp algorithm proposed in [4] to solve (classical) assignment problem for the matrix formed by the rest $2(i - 1)$ lines $(j = n - 2(i - 1) + 1,...,n)$ of i'th layer of matrix C and construct the rest part of transposition π^i according to this solution.

2.2 Correctness and Running Time of Algorithm A

Lemma 1. *Algorithm A is correct for* $m < n$ *and its running time is* $O(mn^2+m^{7/2})$.

Proof. Correctness of the algorithm is determined by existence of a feasible (without infinite elements) solution of (classical) assignment problem in the second part of each stage. As one can see from the description of the algorithm, on i'th stage of it there will be at least $2(i-1)$ unforbidden (finite) elements in each of lines (c_{ij}) for $j = n - 2(i - 1) + 1,...,n$. According to Ore theorem ([6]) it will be enough for the existence of required solution.

As the first part of each stage runs in time $O(n^2)$ and the second part (Hopkroft-Karp algorithm) runs with $O(m^{5/2})$ total running time of all m stages of algorithm A will be $O(mn^2 + m^{7/2})$.

The proof is complete.

3 Probabilistic Analysis of Algorithm A for Random Instances

Let us use the notations presented in [3]. Suppose algorithm A for solving optimization problem. By $F_A(I)$, $F^*(I)$ we denote the values of the objective function when solving an individual problem I by algorithm A and precisely respectively.

Algorithm A *has performance estimates* $(\varepsilon_n, \delta_n)$ in the class K_n of n-dimensional minimization problems if for all n the following inequality is satisfied:

$$\Pr\{F_A(I) > (1 + \varepsilon_n)F^*(I)\} \le \delta_n \tag{1}$$

Here $\Pr\{J\}$ is the probability of a corresponding event j. ε_n is called a *performance ratio*, δ_n is a *fault probability* of algorithm A.

An algorithm is called *asymptotically optimal* in the problem class $K = \bigcup_{i=1}^{\infty} K_n$ if there exist its performance estimates $(\varepsilon_n, \delta_n)$ such that $\varepsilon_n \to 0, \delta_n \to 0$ as $n \to \infty$.

In the paper we consider the following type of random instances. Let all elements c_{ijk} of input matrix C be independent random variables distributed on segment $[a_n, b_n]$ where $b_n > a_n > 0$. Let $m \to \infty$ and $b_n/a_n \to \infty$ as $n \to \infty$.