On the Cycle Polytope of a Directed Graph and Its Relaxations

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1 Introduction

This paper continues the investigation of the cycle polytope of a directed graph begun by Balas and Oosten [2]. Given a digraph $G = (N,A)$ and the collection $\mathcal{C}$ of its simple directed cycles, the cycle polytope defined on G is $P_{\mathcal{C}} := \text{conv}\ \{\chi^C : C \in \mathcal{C}\}$, where χ^C is the incidence vector of C. According to the integer programming formulation given in [2], $P_{\mathcal{C}}$ is the convex hull of points $x \in \mathbb{R}$ satisfying

$$x(\delta^+(i)) - x(\delta^-(i)) = 0 \text{ for all } i \in N \tag{1}$$

$$x(\delta^+(i)) \leq 1 \text{ for all } i \in N \tag{2}$$

$$-x(S,N \setminus S) + x(\delta^+(i)) + x(\delta^+(j)) \leq 1 \text{ for all } S \subseteq N, 2 \leq |S| \leq n-2, \quad i \in S, j \in N \setminus S \tag{3}$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{\pi(i)\pi(j)} \geq 1 \text{ for all permutations } \pi \text{ of } N \tag{4}$$

$$x_{ij} \in \{0,1\} \text{ for all } (i,j) \in A \tag{5}$$

Here $\delta^+(k)$ and $\delta^-(k)$ denote the sets of arcs directed out of and into node k, respectively; for $F \subseteq A$, $x(F) = \sum(x_{ij} : (i,j) \in F)$; and for $S,T \subseteq N$, $x(S,T) := \sum(x_{ij} : i \in S, j \in T)$.

For the rest of this paper, we assume that the digraph G is complete. The dimension of $P_{\mathcal{C}}$ is $(n-1)^2$ (see [2]).

The inequalities of the above formulation, as well as the *nonnegativity constraints* $x_{ij} \geq 0$, were shown in [2] to be facet defining for $P_{\mathcal{C}}$.

Balas and Oosten [2] claimed that the only facet defining inequalities that cut off the origin, i.e. those inequalities $\alpha x \geq \alpha_0$ with $\alpha_0 > 0$, are the *linear ordering constraints* (4) and the inequalities equivalent to them. This claim is false; and

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the main contribution of this paper is to describe the rich family of facet defining inequalities for $P_{\mathcal{C}}$ that cut off the origin and are not equivalent to any inequality (4).

The remainder of this paper is organized as follows: Section 2 examines the relationship between the cycle polytope and some of its relaxations: the cycle cone, the dominant of the cycle polytope, the upper cycle polyhedron. Section 3 deals with the polar relationship between cycles and linear orderings. Section 4 derives a characterization of all the facets of the upper cycle polyhedron. Finally, section 5 focuses on those facet defining inequalities that have coefficients in {0,1,2}.

2 Some Polyhedra Associated with $P_{\mathcal{C}}$

The *dominant* of $P_{\mathcal{C}}$, denoted by $\mathrm{dmt}(P_{\mathcal{C}})$, is the Minkowski sum of the cycle polytope and the nonnegative orthant, i.e. $\mathrm{dmt}(P_{\mathcal{C}}) := P_{\mathcal{C}} + \mathbb{R}^A_+$. Unlike $P_{\mathcal{C}}$, its dominant is full-dimensional, i.e. $\dim(\mathrm{dmt}(P_{\mathcal{C}})) = n(n-1)$.

The *upper cycle polyhedron* $\mathcal{U}_C(G)$ defined on G is $P_{\mathcal{C}} + \mathrm{cone}(C)$, where $\mathrm{cone}(C)$ is the cone generated by its cycles.

Proposition 1.

$$\mathcal{U}_C = dmt(P_{\mathcal{C}}) \cap cone(C).$$

Proof. See Balas and Stephan [3].

It follows from Proposition 1 that $\dim(\mathcal{U}_C) = (n-1)^2$, $P_{\mathcal{C}} \subset \mathcal{U}_C \subset \mathrm{dmt}(P_{\mathcal{C}})$, and $\mathrm{dmt}(P_{\mathcal{C}}) = \mathrm{dmt}(\mathcal{U}_C)$.

Next we address the relationship between valid inequalities and facets of $P_{\mathcal{C}}, \mathcal{U}_C$ and $\mathrm{dmt}(P_{\mathcal{C}})$. First, from $P_{\mathcal{C}} \subseteq \mathcal{U}_C \subseteq \mathrm{dmt}(P_{\mathcal{C}})$ it follows that any inequality valid for $\mathrm{dmt}(P_{\mathcal{C}})$ is also valid for $\mathcal{U}_C$ and $P_{\mathcal{C}}$. Next, from the definition of the dominant, if $\alpha x \geq \alpha_0$ is valid for $\mathrm{dmt}(P_{\mathcal{C}})$, then $\alpha_0 \geq 0$ and $\alpha_{ij} \geq 0$ for all $(i,j) \in A$. As to the relationship between $P_{\mathcal{C}}$ and its dominant, we will make use of a result of Balas and Fischetti [1, Theorem 8]. Applied to our polyhedron, it amounts to the following.

Theorem 1. *Let $\alpha x \geq \alpha_0$, with $\alpha_0 > 0$, be a valid inequality for $\mathrm{dmt}(P_{\mathcal{C}})$ that defines a facet of $P_{\mathcal{C}}$. Then $\alpha x \geq \alpha_0$ is facet inducing for $\mathrm{dmt}(P_{\mathcal{C}})$ if and only if the subgraph of G induced by those arcs (i,j) such that $\alpha_{ij} = 0$ contains a spanning tree.*

From Proposition 1 it follows that, if $\alpha x \geq \alpha_0$ is valid for $\mathcal{U}_C$, then $\alpha x \geq \alpha_0$ is the positive combination of some inequality $\alpha' x \geq \alpha_0$ valid for $\mathrm{dmt}(P_{\mathcal{C}})$ and some equations $x(\delta^+(i)) - x(\delta^-(i)) = 0$, $i \in N$; hence again $\alpha_0 \geq 0$, but it may happen that $\alpha_{ij} < 0$ for some (i,j). However, the next theorem and the two following facts imply that any valid inequality for $\mathcal{U}_C$ (and $P_{\mathcal{C}}$) that cuts off the origin has an equivalent form with all coefficients nonnegative.

Theorem 2. *An inequality $\alpha x \geq \alpha_0$ is valid (is facet defining) for $\mathcal{U}_C$ if and only if $\alpha_0 \geq 0$ and $\alpha x \geq \alpha_0$ is valid (is facet defining) for $P_{\mathcal{C}}$.*

Proof. See Balas and Stephan [3].