Modelling Some Robust Design Problems via Conic Optimization

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Summary. In this paper, we deal with modelling robust design problems via conic optimization. A robust design problem deals with finding a robust optimal solution of an uncertain design problem. The uncertain data is assumed to belong to a so-called uncertainty set $\mathcal{U}$. Uncertainty means that the data is not known exactly at the time when the solution has to be determined. In order to find a robust optimal solution, we use the robust optimization (RO) methodology of Ben-Tal and Nemirovskii. We demonstrate this on the robust shortest path problem (RSPP), the robust maximum flow problem (RMFP) and the robust resistance network topology design (RNTD) problem.

1 Introduction

Conic optimization (CO) is a very useful optimization technique that concerns the problem of minimizing a linear objective function over the intersection of an affine set and a convex cone. The general form of a conic optimization problem is as follows:

$$\min_{x \in \mathbf{R}^n} \{c^T x : Ax - b \in \mathcal{K}\}. \tag{CP}$$

The objective function is $c^T x$, with objective vector $c \in \mathbf{R}^n$. Furthermore, $Ax - b$ represents an affine function from $\mathbf{R}^n$ to $\mathbf{R}^m$, $\mathcal{K}$ denotes a convex cone in $\mathbf{R}^m$ and the constraint matrix A is of size $m \times n$. The importance of conic optimization is due to the fact that it can model many nonlinear optimization problems. Furthermore, conic optimization problems can be solved efficiently using interior-point methods since the theory of self-concordant barriers developed by Nesterov and Nemirovskii [5] provides an algorithm with polynomial complexity.

We demonstrate how to use CO models on three typical problems, i.e., the robust shortest path problem (RSPP), the robust maximum flow problem (RMFP) and the robust resistance network topology design (RNTD) problem.

The paper is organized as follows. In Section 2 we present a short description on modelling robust design problems. The RSPP, the RMFP and the RNTD are presented in Section 3.1, 3.2 and 3.3, respectively. Some concluding remarks are presented in Section 4.

2 Modelling Robust Design Problems via CO

A robust design problem deals with finding a robust optimal solution of an uncertain design problem. The robust optimal solutions of these problems are obtained using the robust optimization (RO) methodology of Ben-Tal and Nemirovskii [1]. In this methodology, it is assumed that the data vector ζ belongs to an uncertainty set $\mathcal{U}$. Thus, an uncertain optimization problem can be expressed as follows.

$$\min\{f_0(x,\zeta)\ :\ f_i(x,\zeta) \le 0, \quad i = 1,\ldots,m\}, \quad \zeta \in \mathcal{U}. \tag{1}$$

One associates with the uncertain problem (1) its so-called robust counterpart which is given by

$$\min\{t\ :\ f_0(x,\zeta) \le t, \quad f_i(x,\zeta) \le 0, \quad i = 1,\ldots,m, \quad \forall \zeta \in \mathcal{U}\}. \tag{2}$$

Obviously, this robust counterpart of (1) represents a worst-case oriented approach: a pair of solutions $(t; x)$ is feasible only if x satisfies the constraints in (1) for all possible values of ζ. The optimal solutions of (2) are called robust optimal solutions. The robust counterpart (2) is an optimization problem with usually infinitely many constraints, depending on the uncertainty set $\mathcal{U}$, thus this problem may be very hard to solve. This means that only if $\mathcal{U}$ is chosen suitably, the problem (2) can be solved efficiently.

We consider the uncertain optimization problem in conic form:

$$\min_{x\in\mathbf{R}^n}\{c^T x\ :\ A_i x - b_i \in \mathcal{K}_i, i = 1,\ldots,m\}, \quad (c,\{A_i,b_i\}_{i=1}^m) \in \mathcal{U} \tag{3}$$

where for every i, $\mathcal{K}_i$ is a convex cone and where $\mathcal{U}$ is the uncertainty set. We restrict ourselves to the cases when each $\mathcal{K}_i$ is either the nonnegative orthant, or the Lorentz (second-order or ice cream) cone, or the positive semidefinite cone. Thus the robust counterpart to (3) can be expressed in the following CO model:

$$\min\left\{t\ :\ c^T x - t \le 0, A_i x - b_i \in \mathcal{K}_i, i = 1,\ldots,m, \forall (c,\{A_i,b_i\}_{i=1}^m) \in \mathcal{U}\right\}. \tag{4}$$

3 Applications on Some Robust Design Problems

In this section we present the CO models of the three robust design problems mentioned before, i.e., the RSPP, the RMFP and the RNTD.

3.1 Robust Shortest Path Problem (RSPP)

Based on [3], in this section we discuss the problem of finding the robust shortest path. For a given network $G = (\mathcal{V}, \mathcal{A})$ with node set $\mathcal{V}$ and arc set $\mathcal{A}$, an instance of the SPP consists of finding a directed path from a single node s to a single node t with a minimal total length with respect to a given length function $c : \mathcal{A} \to \mathbf{R}_+$. A binary optimization model for the shortest path problem is

$$\min\{c^T x\ :\ Ax = b, x \in \{0,1\}^{\mathcal{A}}\}, \tag{5}$$