Parameter Estimation for Stock Models with Non-Constant Volatility Using Markov Chain Monte Carlo Methods

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Summary. We consider a model for a financial market where the asset prices satisfy a stochastic differential equation. For the volatility no new source of randomness is introduced, but the volatility at each time depends deterministically on all previous price fluctuations. Such non-constant volatility models preserve the completeness of the market while they allow for many attractive features.

We deal with the estimation of the occurring parameters given discretely observed data deriving a suitable Markov chain Monte Carlo framework. We illustrate this with an example dealt with in [2] where the drift term is a continuous time Markov chain with finitely many states (which cannot be observed directly) and the volatility is given by the Hobson-Rogers model [3]. We also present numerical results.

1 Introduction

For a finite time horizon $T > 0$ we look at a stock market model. The stock returns are given by $dR_t = \mu_t dt + \sigma_t dW_t$, where the drift process $\mu = (\mu_t)_{t\in[0,T]}$ is independent of the Brownian motion W. For the volatility $\sigma = (\sigma_t)_{t\in[0,T]}$ we consider so-called non-constant processes that are functions of observable factor processes, i.e. they depend deterministically on the past stock prices, and hence introduce no new source of randomness. Thus we have a complete market which is not true for (fully) stochastic volatility models where a non-traded source of risk is included. We describe a very general and flexible framework to calibrate such models to market data using Markov chain Monte Carlo (MCMC) methods popular in finance since [4].

One example used as an illustration is the Hobson-Rogers (HR) model introduced in [3]. There the volatility is a deterministic function of the past underlying prices represented by finitely many offset functions. This model accounts for the possibility of volatility smiles and skews as well as term structures, even using a model based

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on the first offset function only. In [5] it is found that the fit of the HR model is generally better than that of the Heston model.

2 A Stock Model with Non-Constant Volatility

Let $(\Omega,\mathcal{A},P)$ be a complete probability space, $T > 0$ the terminal trading time, and $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration in $\mathcal{A}$ satisfying the usual conditions, i.e. $\mathcal{F}$ is right-continuous and contains all P-null sets. We consider a riskless asset (bond) with non-negative interest rates $(r_t)_{t\in[0,T]}$ and n risky assets (stocks). The dynamics of the n-dimensional price process $S = (S_t)_{t\in[0,T]}$ of the stocks is given by

$$dS_t = \mathrm{Diag}(S_t)(\mu_t dt + \sigma_t dW_t) \ , \quad S_0^i > 0 \ , \tag{1}$$

where $W = (W_t)_{t\in[0,T]}$ is an n-dimensional Brownian motion w.r.t. $\mathcal{F}$. The drift process $\mu = (\mu_t)_{t\in[0,T]} \in \mathbb{R}^n$ is independent of W, the volatilities $(\sigma_t)_{t\in[0,T]} \in \mathbb{R}^{n\times n}$ are non-singular, and both μ and σ are progressively measurable w.r.t. $\mathcal{F}$. The return process $R = (R_t)_{t\in[0,T]}$ associated with the stock is defined by $dR_t = \mathrm{Diag}(S_t)^{-1}dS_t$, the excess return process is defined by $d\tilde{R}_t = dR_t - r_t dt$.

Without loss of generality we can assume σ_t to be a lower triangular matrix with positive diagonal. Since $\sigma_t\sigma_t^\top dt = d[R]_t$ and σ_t is given uniquely as the square root, $(\sigma_t)_{t\in[0,T]}$ is adapted to $\mathcal{F}^S$ as well as to $\mathcal{F}^R$. Here $[\,\cdot\,]$ denotes the quadratic variation, and $\mathcal{F}^X = (\mathcal{F}_t^X)_{t\in[0,T]}$ denotes the filtration of augmented σ-algebras generated by the $\mathcal{F}$-adapted process $X = (X_t)_{t\in[0,T]}$.

Next, we introduce the risk neutral probability measure $\tilde{P}$. The density process is defined by $dZ_t = -\sigma_t^{-1}(\mu_t - r_t)Z_t dW_t$, $Z_0 = 1$. We assume that Z is a martingale and define $\tilde{P}$ by $d\tilde{P} = Z_T dP$. Girsanov's theorem guarantees that $\tilde{W}_t = W_t + \int_0^t \sigma_s^{-1}(\mu_s - r_s)ds$ is a $\tilde{P}$-Brownian motion. The excess return process is a martingale under $\tilde{P}$ satisfying $d\tilde{R}_t = \sigma_t d\tilde{W}_t$.

We model the dynamics of the volatilities under $\tilde{P}$ using an $\mathcal{F}^S$-adapted process ξ which leads to a Markovian structure introducing the m-dimensional factor process $\xi = (\xi_t)_{t\in[0,T]}$ with dynamics

$$d\xi_t = \nu(\xi_t)dt + \tau(\xi_t)d\tilde{W}_t \ , \tag{2}$$

where ν and τ are $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$-valued. The volatility process takes the form $\sigma_t = f_\sigma(\xi_t)$. We demand that ν, τ, and f_σ are measurable.

Remark 1. In (2) no new Brownian motion is introduced for the volatility model. Under standard Lipschitz and linear growth conditions on ν, τ, f_σ the system consisting of $d\tilde{R}_t = \sigma_t d\tilde{W}_t$ and (2) has a strong solution $(\tilde{R}, \xi)$ (which is Markovian) and the model is complete w.r.t. $\mathcal{F}_T^S$ (cf. [2]).

3 A General MCMC Framework

We construct a block-wise Metropolis-Hastings (MH) sampler for the time-discretised model. To this end we have to augment the unknowns such that the complete-data likelihood (CDLH) is computable, fix prior distributions, and construct update schemes. Here we employ the original measure P.