Complexity of Pure-Strategy Nash Equilibria in Non-Cooperative Games

Juliane Dunkel

Operations Research Center, Massachusetts Institute of Technology, U.S.A.
juliane@mit.edu

1 Introduction

Game theory in general and the concept of Nash equilibrium in particular have lately come under increased scrutiny by theoretical computer scientists. Computing a mixed Nash equilibrium is a case in point. For many years, one of the most important open problems was the complexity of computing a mixed Nash equilibrium in games with only two players. Only recently was it solved by a sequence of significant papers (Goldberg and Papadimitriou (2006), Daskalakis et.al. (2006), Chen and Deng (2005), Daskalakis and Papadimitriou (2005), and Chen and Deng (2006)).

While Nash (1951) showed that mixed Nash equilibria do exist in any finite noncooperative game, it is well known that pure-strategy Nash equilibria are in general not guaranteed to exist. It is therefore natural to ask which games have pure-strategy Nash equilibria and, if applicable, how difficult is it to find one. In this article, we study these questions for certain classes of weighted congestion and local-effect games.

Congestion games are a fundamental class of noncooperative games possessing pure-strategy Nash equilibria. Their wide applicability (e.g., in network routing and production planning) has made them an object of extensive study. In the network version, each player wants to route one unit of flow on a path from her origin to her destination at minimum cost, and the cost of using an arc only depends on the total number of players using that arc. A natural extension is to allow for players sending different amounts of flow, which results in so-called weighted congestion games. While examples have been exhibited showing that pure-strategy Nash equilibria need not exist, we prove that it actually is NP-hard to determine whether a given weighted network congestion game has a pure-strategy Nash equilibrium. This is true regardless of whether flow is unsplittable (has to be routed on a single path for each player) or not.

A related family of games are local-effect games, where the disutility of a player taking a particular action depends on the number of players taking the same action and on the number of players choosing related actions. We show that the problem of deciding whether a bidirectional local-effect game has a pure-strategy Nash equilibrium is NP-complete, and that the problem of finding a pure-strategy Nash

equilibrium in a bidirectional local-effect game with linear local-effect functions (for which the existence of a pure-strategy Nash equilibrium is guaranteed) is PLS-complete. The latter proof uses a tight PLS-reduction, which implies the existence of instances and initial states for which any sequence of selfish improvement steps needs exponential time to reach a pure-strategy Nash equilibrium.

Due to space limitations, proofs are only sketched or omitted completely. Details can be found in Dunkel (2005).

2 Weighted Congestion Games

In an *unweighted congestion game*, we are given a set of players $N = \{1, 2, \ldots, n\}$, and a set of resources E. For each player $i \in N$, her set S_i of available strategies is a collection of subsets of the resources. A cost function $f_e : \mathbb{N} \to \mathbb{R}_+$ is associated with each resource $e \in E$. Given a strategy profile $s = (s_1, s_2, \ldots, s_n) \in S = S_1 \times S_2 \times \cdots \times S_n$, the cost of player i is $c_i(s) = \sum_{e \in s_i} f_e\big(n_e(s)\big)$, where $n_e(s)$ denotes the number of players using resource e in s. In other words, in a congestion game each player chooses a subset of resources that are available to her, and the cost to a player is the sum of the costs of the resources used by her, where the cost of a resource only depends on the total number of players using this resource.

In a *network congestion game*, the arcs of an underlying directed network represent the resources. Each player $i \in N$ has an origin-destination pair (a_i,b_i) of nodes, and the set S_i of pure strategies available to player i is the set of directed (simple) paths from a_i to b_i. In a symmetric network congestion game all players have the same origin-destination pair.

In a *weighted network congestion game*, each player $i \in N$ has a positive integer weight w_i, which constitutes the amount of flow she wants to ship from a_i to b_i. In the case of unsplittable flows, the cost of player i adopting strategy s_i in a strategy profile $s = (s_1, s_2, \ldots, s_n) \in S$ is given by $c_i(s) = \sum_{e \in s_i} f_e\big(\theta_e(s)\big)$, where $\theta_e(s) = \sum_{i: e \in s_i} w_i$ denotes the total flow on arc e in s. In integer-splittable network congestion games, a player with weight greater than one can choose a subset of paths on which to route her flow simultaneously.

While every unweighted congestion game possesses a pure-strategy Nash equilibrium (Rosenthal 1973), this is not true for weighted congestion games; see, e.g., Fig. 1 in Fotakis, Kontogiannis, and Spirakis (2005). We can actually turn their instance into a gadget to derive the following result.

Theorem 1. *The problem of deciding whether a weighted symmetric network congestion game with unsplittable flows possesses a pure-strategy Nash equilibrium is NP-complete.*

For network congestion games with integer-splittable flows, we obtain the following result.

Theorem 2. *The problem of deciding whether a weighted network congestion game with integer-splittable flows possesses a pure-strategy Nash equilibrium is strongly NP-hard. Hardness even holds if there is only one player with weight* 2*, and all other players have unit weights.*