Traffic Optimization Under Route Constraints with Lagrangian Relaxation and Cutting Plane Methods

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Summary. The optimization of traffic flow in congested urban road networks faces a well-known dilemma: Optimizing system performance is unfair with respect to the individual drivers' travel times; and a fair user equilibrium may result in bad system performance. As a remedy, computing a system optimum with fairness conditions, realized by length constraints on the routes actually used by drivers, has been suggested in [5]. This poses interesting mathematical challenges, namely the non-linearity of the objective function and the necessity to deal with path constraints in large networks. While the authors present results suggesting that solutions to this constrained system optimum problem (CSO) are indeed equally good and fair, they rely on a standard Frank-Wolfe/Partan-algorithm to obtain them.

In this paper, we present a Lagrangian relaxation of the CSO problem for which the Lagrangian dual function can be evaluated by a decomposition into constrained shortest path problems which we solve exactly employing state-of-the-art acceleration techniques. The Lagrangian dual problem is then solved by a special cutting plane method.

Finally, we obtain test results which suggest that this approach outperforms previously described solution schemes for the CSO problem.

1 The Constrained System Optimum Problem

The street network is represented by a digraph $G = (V, A)$ where each arc is equipped with an increasing differentiable latency function $l_a : \mathbb{R}_+ \to \mathbb{R}_+$ associating a latency time $l_a(x_a)$ with a given flow rate x_a on arc a. We use the following latency function with free flow travel time l_a° and tuning parameters α and β suggested by the U. S. Bureau of Public Roads:

$$l_a(x_a) := l_a^\circ \left(1 + \alpha \left(\frac{x_a}{c_a}\right)^\beta\right).$$

Additionally, we define a "normal length" τ_a for each arc which will be used to control the fairness of the calculated route suggestions. Technically, τ_a may be any

metric on A; in order to achieve equally good and fair routes, we follow the suggestion of the authors of [5] and choose τ_a to be the latency incurred by drivers on arc a in a user equilibrium.

The set K of traffic demands is represented by origin destination pairs $(s_k, t_k) \in V \times V$ with an associated demand d_k for each $k \in K$. Let P_k represent the set of all possible paths for commodity k. Let $P := \cup_{k \in K} P_k$ denote the set of all paths in G possibly used and $\tau_p := \sum_{a \in p} \tau_a$ the normal travel time incurred on $p \in P$.

The parameter $\varphi \geq 1$ defines the feasibility of routes: a path $p \in P_k$ is feasible iff $\tau_p \leq \varphi T_k$, $T_k := min_{p \in P_k} \tau_p$ being the length of a shortest s_k-t_k path according to normal length. In a certain sense, φ represents the factor by which a driver's travel time may increase in our solution compared to the user equilibrium.

With z_a^k representing the flow rate of commodity k on arc a and x_p the flow rate on path p, the CSO problem may be stated as the following non-linear multi-commodity flow problem with path constraints:

$$
\begin{aligned}
&\textit{Minimize} && \sum_{a \in A} l_a(x_a) x_a && && \text{(1a)}\\
&\textit{subject to} && \sum_{k \in K} z_a^k = x_a && a \in A && \text{(1b)}\\
& && \sum_{p \in P_k : a \in p} x_p = z_a^k && a \in A;\ k \in K && \text{(1c)}\\
& && \sum_{p \in P_k} x_p = d_k && k \in K && \text{(1d)}\\
& && \tau_p \leq \varphi T_k && p \in P_k : x_p > 0;\ k \in K && \text{(1e)}\\
& && x_p \geq 0 && p \in P. &&
\end{aligned}
$$

Note that the objective (1a) is convex; furthermore, the convex set of feasible solutions may be artificially bounded by $x_a \leq D_k := \sum_{k \in K} d_k$ and thus be made compact.

2 Lagrangian Relaxation for the CSO Problem

We now state a Lagrangian relaxation of (1) in which we drop constraints (1b) coupling the flow rates for each commodity z_a^k with the total arc flows x_a. A similar idea is used in [2] to solve different variants of the multi-commodity flow problem. With Lagrangian multipliers u_a, the resulting relaxation for the CSO problem is