Trading Regions Under Proportional Transaction Costs

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Summary. In the Black-Scholes model optimal trading for maximizing expected power utility under proportional transaction costs can be described by three intervals B, NT, S: If the proportion of wealth invested in the stocks lies in B, NT, S, then buying, not trading and selling, respectively, are optimal. For a finite time horizon, the boundaries of these trading regions depend on time and on the terminal condition (liquidation or not). Following a stochastic control approach, one can derive parabolic variational inequalities whose solution is the value function of the problem. The boundaries of the active sets for the different inequalities then provide the boundaries of the trading regions. We use a duality based semi-smooth Newton method to derive an efficient algorithm to find the boundaries numerically.

1 Trading Without Transaction Costs

The continuous-time Black Scholes model consists of one bond or bank account and one stock with prices $(P_0(t))_{t\in[0,T]}$ and $(P_1(t))_{t\in[0,T]}$ which for interest rate $r \geq 0$, trend $\mu \in \mathbb{R}$, and volatility $\sigma > 0$ evolve according to

$$dP_0(t) = P_0(t)\, r\, dt\ , \quad dP_1(t) = P_1(t)\,(\mu\, dt + \sigma\, dW(t))\, , \quad P_0(0) = P_1(0) = 1\ ,$$

where $W = (W(t))_{t\in[0,T]}$ is a Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ denote the augmented filtration generated by W.

Without transaction costs the trading of an investor may be described by initial capital $\underline{x} > 0$ and risky fraction process $(\eta(t))_{t\in[0,T]}$, where $\eta(t)$ is the fraction of the portfolio value (wealth) which is held in the stocks at time t. The corresponding *wealth process* $(X(t))_{t\in[0,T]}$ is defined self-financing by

$$dX(t) = (1-\eta(t))X(t)r\, dt + \eta(t)\, X(t)\,(\mu\, dt + \sigma\, dW(t))\, , \quad X(0) = \underline{x}\ .$$

The utility of terminal wealth $x > 0$ is given by power utility $\frac{1}{\alpha}\, x^\alpha$ for any $\alpha < 1$, $\alpha \neq 0$. The parameter α models the preferences of an investor. The limiting case $\alpha \to 0$ corresponds to logarithmic utility i.e. maximizing the expected rate of return,

$\alpha > 0$ corresponds to less risk averse and $\alpha < 0$ to more risk averse utility functions. Merton showed that for logarithmic ($\alpha = 0$) and power utility the optimal trading strategy is given by a constant optimal risky fraction

$$\eta(t) = \hat{\eta}\,, \quad t \in [0,T], \quad \text{where} \quad \hat{\eta} = \frac{1}{1-\alpha}\,\frac{\mu - r}{\sigma^2}\,. \tag{1}$$

2 Proportional Transaction Costs

To keep the risky fraction constant like in (1) involves continuous trading which, under transaction costs, is no longer adequate. For possible cost structures see e.g. [5, 7, 8]. We consider proportional costs $\gamma \in (0,1)$ corresponding to the proportion of the traded volume which has to be paid as fees.

For suitable infinite horizon criteria solution of the corresponding Hamilton-Jacobi-Bellmann equation (HJB) leads to a characterization of the optimal wealth process as a diffusion reflected at the boundaries of a cone, see [3, 9]. When reaching the boundaries of the cone, infinitesimal trading occurs in such a way that the wealth process just stays in the cone. The cone corresponds to an interval for the risky fraction process. The existence of a viscosity solution for the HJB equation for finite time horizon is shown in [1] and numerically treated in [10] using a finite difference method.

Now let us fix costs $\gamma \in (0,1)$ and parameters $\alpha < 1$, $\alpha \neq 0$, r, μ, σ such that $\hat{\eta} \in (0,1)$. The trading policy can be described by two increasing processes $(L(t))_{t\in[0,T]}$ and $(M(t))_{t\in[0,T]}$ representing the cumulative purchases and sales of the stock. We require that these are right-continuous, $\mathcal{F}$-adapted, and start with $L(0-) = M(0-) = 0$. Transaction fees are paid from the bank account. Thus the dynamics of the controlled wealth processes $(X_1(t))_{t\in[0,T]}$ and $(X_0(t))_{t\in[0,T]}$, corresponding to the amount of money on the bank account and the amount invested in the stocks, are

$$\begin{aligned} dX_0(t) &= rX_0(t)\,dt - (1+\gamma)\,dL(t) + (1-\gamma)\,dM(t)\,, \\ dX_1(t) &= \mu X_1(t)\,dt + \sigma X_1(t)\,dW(t) + dL(t) - dM(t)\,. \end{aligned}$$

The objective is the maximization of expected utility at the terminal trading time T, now over all control processes $(L(t))_{t\in[0,T]}$ and $(M(t))_{t\in[0,T]}$ which satisfy the conditions above and for which the wealth processes X_0 and X_1 stay positive and the total wealth strictly positive i.e. $(X_0(t), X_1(t)) \in \mathcal{D} := I\!R_+^2 \setminus \{(0,0)\}$, $t \in [0,T]$. So suppose $(\underline{x}_0, \underline{x}_1) = (X_0(0-), X_1(0-)) \in \mathcal{D}$. We distinguish the maximization of expected utility for the terminal total wealth,

$$\tilde{J}(t,x_0,x_1) = \sup_{(L,M)} \mathrm{E}[\tfrac{1}{\alpha}\,(X_0(T) + X_1(T))^{\alpha} \,|\, X_0(t) = x_0, X_1(t) = x_1]\,,$$

and of the terminal wealth after liquidating the position in the stocks,

$$J(t,x_0,x_1) = \sup_{(L,M)} \mathrm{E}[\tfrac{1}{\alpha}\,(X_0(T) + (1-\gamma)X_1(T))^{\alpha} \,|\, X_0(t) = x_0, X_1(t) = x_1]\,.$$

We always assume $\tilde{J}(0,x_0,x_1) < \infty$ for all $(x_0,x_1) \in \mathcal{D}$.