The Markov-Modulated Risk Model with Investment

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Summary. We consider Markov-modulated risk reserves which can be invested into a stock index following a geometric Brownian motion. Within a special class of investment policies we identify one which maximizes the adjustment coefficient. A comparison to the compound Poisson case is also given.

1 Introduction and Model

In this paper we combine two features of risk models which have so far only been investigated separately: Markov-modulation (cf. [1]) and maximizing the adjustment coefficient by optimal investment (cf. [6]). In the Markov-modulated Poisson model the premium rate and claim arrivals are determined by a Markov-modulated environment which is described by an irreducible continuous-time Markov process J defined on some finite state space $E = \{1, \ldots ,d\}$ with intensity matrix $Q = (q_{ij})_{i,j\in E}$. If not stated otherwise, the distribution of J_0 is arbitrary. We only suppose that $\mathbb{P}(J_0 = i) > 0$ for all $i \in E$. J_t can be interpreted as the general economic conditions which are present at time t. J_t influences the premium rate, the arrival intensity of claims and the claim size distribution as follows: The premium income rate at time t is c_{J_t}, i.e. as long as $J_t = i$ we have a linear income stream at rate c_i. Claim arrivals are according to a Poisson process with rate λ_{J_t}. Thus, $N = \{N_t, t \geq 0\}$ is a Markov-modulated Poisson process. A claim U_k which occurs at time t has distribution B_{J_t}, where B_i is some distribution concentrated on $(0,\infty)$. As usual the claim sizes $U_1, U_2, \ldots$ are assumed to be conditionally independent given J and μ_i is the finite expectation of B_i.

In our model the insurer has the opportunity to invest into a stock index or say some portfolio whose price process $S := \{S_t,t \geq 0\}$ is modeled by a geometric Brownian motion with dynamics $dS_t = S_t\,(a\,dt + b\,dW_t)$ with $a \in \mathbb{R}$, $b > 0$. K_t denotes the amount of money which the insurer invests into the portfolio at time t and is also allowed to be negative or even larger than the actual wealth for any $t \geq 0$. This fact can be interpreted as the possibility to sell the portfolio short or to borrow an arbitrary amount of money from the bank respectively. The remaining

part of the insurers reserve is invested into a bond which yields no interest. Our aim is to maximize the adjustment coefficient, i.e. the parameter which determines the exponential decay of the ruin probability w.r.t. the initial reserve. In this paper we restrict ourselves to the case where the admissible investment strategies depend on the environment only, i.e. we consider functions $k : E \to \mathbb{R}$ such that the investment strategy K is given by $K_t = k(J_t)$. This is reasonable since in the paper by [6] the authors show in the model without Markov-modulation that a constant investment strategy maximizes the adjustment coefficient. Thus, we suppose that the wealth process of the insurer is given by

$$Y_t(u,K) = u + \int_0^t c_{J_s}\, ds - \sum_{k=1}^{N_t} U_k + \int_0^t \frac{K_v}{S_v}\, dS_v\,. \tag{1}$$

By applying the time change $\hat{Y}_t := Y_{T(t)}$ with $T(t) := \int_0^t \frac{1}{c_{J_s}}\, ds$ the ruin probability does not change and we can w.l.o.g. assume that $c(\cdot) \equiv c \in \mathbb{R}$.

In this paper we suppose that all claims have exponential moments, i.e. there exists a possibly infinite constant $r_\infty^{(i)} > 0$ such that the centered moment generating function $h_i(r) := \int_0^\infty e^{rx}\, dB_i(x) - 1$ is finite for every $r < r_\infty^{(i)}$. It is moreover assumed that $h_i(r) \to \infty$ as $r \to r_\infty^{(i)}$. This assumption implies that h_i is increasing, convex and continuous on $[0, r_\infty^{(i)})$ with $h_i(0) = 0$. An important part of this assumption is that $h_i(r) < \infty$ for some $r > 0$. Thus, the tail of the distribution B_i decreases at least exponentially fast.

When applying an investment strategy K, the ruin probability in infinite time is then for $u \geq 0$ defined by

$$\Psi(u,K) = \mathbb{P}\Big(\inf_{t\geq 0} Y_t(u,K) < 0\Big)\,.$$

If $\tau(u,K) := \inf\{t > 0\,; Y_t(u,K) < 0\}$ is the time of ruin, then $\Psi(u,K) = \mathbb{P}(\tau(u,K) < \infty)$. If we denote by $\pi = (\pi)_{i\in E}$ the stationary distribution of J (which exists and is unique since J is irreducible and has finite state space) then $\rho^{(K)} := c + a \sum_{i\in E} \pi_i\, k(i) - \sum_{i\in E} \pi_i \lambda_i \mu_i$ for some investment strategy $K = k(J)$ is the difference between the average premium income and the average payout when using the investment strategy $K = k(J)$. We refer to $\rho^{(K)}$ as the safety loading with respect to the investment strategy K.

Lemma 1. *Let $K = k(J)$ and suppose that $\rho^{(K)} \leq 0$. Then for all $u \geq 0$ it holds that $\Psi(u,K) = 1$.*

The proof of this statement is omitted since it is standard. For the remaining sections we assume that $\rho^{(K)} > 0$.

2 Fixed Investment Strategies

For a given investment strategy $K = k(J)$ our aim is to find a constant $R^{(K)} > 0$ such that for all $\varepsilon > 0$:

$$\lim_{u\to\infty} \Psi(u,K) e^{(R^{(K)}-\varepsilon)u} = 0\,, \text{ and } \lim_{u\to\infty} \Psi(u,K) e^{(R^{(K)}+\varepsilon)u} = \infty\,. \tag{2}$$