Optimal Portfolios Under Bounded Shortfall Risk and Partial Information

Ralf Wunderlich[1], Jörn Sass[2] and Abdelali Gabih[3]

[1] Mathematics Group,Zwickau University of Applied Sciences, Germany
ralf.wunderlich@fh-zwickau.de
[2] RICAM, Austrian Academy of Sciences, Linz, Austria
joern.sass@oeaw.ac.at
[3] Dept. of Mathematics and Computer Science, Leipzig University, Germany,
abdelali.gabih@math.uni-leipzig.de

1 Introduction

This paper considers the optimal selection of portfolios for utility maximizing investors under a shortfall risk constraint for a financial market model with partial information on the drift parameter. It is known that without risk constraint the distribution of the optimal terminal wealth often is quite skew. In spite of its maximum expected utility there are high probabilities for values of the terminal wealth falling short a prescribed benchmark. This is an undesirable and unacceptable property e.g. from the viewpoint of a pension fund manager. While imposing a strict restriction to portfolio values above a benchmark leads to considerable decrease in the portfolio's expected utility, it seems to be reasonable to allow shortfall and to restrict only some shortfall risk measure.

A very popular risk measure is value at risk (VaR) which takes into account the probability of a shortfall but not the actual size of the loss. Therefore we use the so-called expected loss criterion resulting from averaging the magnitude of the losses. And in fact, e.g. in Basak, Shapiro [1] it is shown that the distribution of the resulting optimal terminal wealth has more desirable properties.

We use a financial market model which allows for a non-constant drift which is not directly observable. In particular, we use a hidden Markov model (HMM) where the drift follows a continuous time Markov chain. In [13] it was shown that on market data utility maximizing strategies based on such a model can outperform strategies based on the assumption of a constant drift parameter. Extending these results to portfolio optimization problems under risk constraints we obtain in Theorem 2 and 3 quite explicit representations for the form of the optimal terminal wealth and the trading strategies which can be computed using Monte Carlo methods.

For additional topics such as stochastic interest rates, stochastic volatiliy, motivation of the model, Malliavin calculus, aspects of parameter estimation, and for more references concerning partial information see [8, 9, 13]. For an control theoretic approach see Rieder, Bäuerle [12].

For more background, references and results on optimization under risk constraints see e.g. Basak, Shapiro [1], Gundel, Weber [7], Lakner, Nygren [11], and [3, 4, 5].

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2 An HMM for the Stock Returns

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, $T > 0$ the terminal trading time, and $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration in $\mathcal{A}$ satisfying the usual conditions. We consider one *money market* with interest rates equal 0 (to simplify notation) and n stocks whose *prices* $S = (S_t)_{t\in[0,T]}$, $S_t = (S_t^1, \dots ,S_t^n)^\top$ evolve according to

$$dS_t = \mathrm{Diag}(S_t)(\mu_t\, dt + \sigma\, dW_t)\ , \quad S_0 \in I\!R^n\ ,$$

where $W = (W_t)_{t\in[0,T]}$ is an n-dimensional Brownian motion w.r.t. $\mathcal{F}$, and σ is the non-singular $(n \times n)$*-volatility-matrix*. The *return process* $R = (R_t)_{t\in[0,T]}$ is defined by $dR_t = (\mathrm{Diag}(S_t))^{-1}\, dS_t$.We assume that $\mu = (\mu_t)_{t\in[0,T]}$, the *drift process* of the return, is given by $\mu_t = B\, Y_t$, where $Y = (Y_t)_{t\in[0,T]}$ is a stationary, irreducible, *continuous time Markov chain* independent of W with state space $\{e_1, \dots ,e_d\}$, the standard unit vectors in $I\!R^d$. The columns of the *state matrix* $B \in I\!R^{n\times d}$ contain the d possible states of μ_t. Further Y is characterized by its *rate matrix* $Q \in I\!R^{d\times d}$, where $\lambda_k = -Q_{kk} = \sum_{l=1,l\neq k}^{d} Q_{kl}$ is the rate of leaving e_k and Q_{kl}/λ_k is the probability that the chain jumps to e_l when leaving e_k.

Since the *market price of risk*, $\vartheta_t = \sigma^{-1}\mu_t = \sigma^{-1}B\, Y_t$, $t \in [0,T]$, is uniformly bounded the density process $(Z_t)_{t\in[0,T]}$ defined by $dZ_t = -Z_t\vartheta_t^\top\, dW_t$, $Z_0 = 1$, is a martingale. By $d\widetilde{P} = Z_T dP$ we define the *risk-neutral* probability measure. $\widetilde{\mathrm{E}}$ will denote expectation with respect to $\widetilde{P}$. Girsanov's Theorem guarantees that $d\widetilde{W}_t = dW_t + \vartheta_t\, dt$ defines a $\widetilde{P}$-Brownian motion. The definition of R yields

$$R_t = \int_0^t B\, Y_s\, ds + \sigma\, W_t = \sigma\, \widetilde{W}_t\ , \quad t \in [0,T]\ . \tag{1}$$

We consider the case of *partial information* meaning that an investor can only observe the prices. Neither the drift process nor the Brownian motion are observable. Only the events of $\mathcal{F}^S$, the augmented filtration generated by S, can be observed and hence all investment decisions have to be adapted to $\mathcal{F}^S$. Note that $\mathcal{F}^S = \mathcal{F}^R = \mathcal{F}^{\widetilde{W}}$.

A *trading strategy* $\pi = (\pi_t)_{t\in[0,T]}$ is an n-dimensional $\mathcal{F}^S$-adapted, measurable process which satisfies $\int_0^T \|\pi_t\|^2 dt < \infty$. The wealth invested in the i-th stock at time t is π_t^i and $X_t^\pi - \mathbf{1}_n^\top\pi_t$ is invested in the money market, where $(X_t^\pi)_{t\in[0,T]}$ is the corresponding *wealth process*. For *initial capital* $x_0 > 0$ it is defined by $dX_t^\pi = \pi_t^\top(\mu_t\, dt + \sigma\, dW_t)$, $X_0^\pi = x_0$. A trading strategy π is called *admissible* if $P(X_t^\pi \geq 0$ for all $t \in [0,T]) = 1$. By Itô's rule

$$X_t^\pi = x_0 + \int_0^t \pi_s^\top \sigma\, d\widetilde{W}_s\ , \quad t \in [0,T]\ . \tag{2}$$

A *utility function* $U : [0,\infty) \to I\!R\cup\{-\infty\}$ is strictly increasing, strictly concave, twice continuously differentiable, and satisfies the Inada conditions $\lim_{x\to\infty} U'(x) = 0$,