Multistage Stochastic Programming Problems; Stability and Approximation

Vlasta Kanková

Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Praha, Czech Republic
kankova@utia.cas.cz

1 Introduction

A multistage stochastic programming problem can be introduced as a finite system of parametric one–stage optimization problems with an inner type of dependence and mathematical (mostly conditional) expectation in objective functions of the individual problems (for more details see e.g. [1], [3], [8]). The constraints sets can depend on the "underlying" probability measure.

To recall the multistage (M–stage, $M \geq 2$) stochastic programming problem let $\xi^j := \xi^j(\omega)$, $j = 1, \ldots, M$ denote an s–dimensional random vector defined on a probability space $(\Omega, \mathcal{S}, P)$; $F^{\xi^j}(z^j)$, $z^j \in R^s$, $j = 1, 2 \ldots, M$ denote the distribution function of the ξ^j and $F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})$, $z^k \in R^s$, $\bar{z}^{k-1} \in R^{(k-1)s}$, $k = 2, \ldots, M$ denote the conditional distribution function (ξ^k conditioned by $\bar{\xi}^{k-1}$); $Z_{F^{\xi^j}} \subset R^s$, $j = 1, 2, \ldots, M$ denote the support of the probability measure corresponding to F^{ξ^j}. Let, moreover, $g_0^M(\bar{x}^M, \bar{z}^M)$ be a continuous function defined on $R^{nM} \times R^{sM}$; $X^k \subset R^n$, $k = 1, 2, \ldots, M$ be a nonempty set; $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) := \mathcal{K}^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k, \bar{z}^k)$, $k = 1, \ldots, M-1$ be a multifunction mapping $R^{nk} \times R^{sk}$ into the space of subsets of R^n. $\bar{\xi}^k (:= \bar{\xi}^k(\omega)) = [\xi^1, \ldots, \xi^k]$; $\bar{z}^k = [z^1, \ldots, z^k]$, $z^j \in R^s$; $\bar{x}^k = [x^1, \ldots, x^k]$, $x^j \in R^n$; $\bar{X}^k = X^1 \times X^2 \ldots \times X^k$; $\bar{Z}^k_{\mathcal{F}} = Z_{F^{\xi^1}} \times Z_{F^{\xi^2}} \ldots \times Z_{F^{\xi^k}}$, $j = 1, \ldots, k$, $k = 1, 2, \ldots, M$. See that (generally) the multifunctions $\mathcal{K}^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k, \bar{z}^k)$, $k = 1, \ldots, M-1$ can depend on the system of the conditional probability measures.

We introduce the M–stage stochastic programming problem ($M \geq 2$) in a rather general form as the problem.

Find
$$\varphi_{\mathcal{F}}(M) = \inf \{\mathsf{E}_{F^{\xi^1}} g^1_{\mathcal{F}}(x^1, \xi^1) |\ x^1 \in \mathcal{K}^1\}, \tag{1}$$

where the function $g^1_{\mathcal{F}}(x^1, z^1)$ is defined recursively

$$
\begin{aligned}
g^k_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) \quad &= \inf\{\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}\; g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1},\, \bar{\xi}^{k+1})\,|x^{k+1} \in \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k)\},\\
k = 1,\, \ldots,\, M-1,\\
g^M_{\mathcal{F}}(\bar{x}^M,\, \bar{z}^M) \quad &:= g^M_0(\bar{x}^M,\, \bar{z}^M), \quad \mathcal{K}^1 = X^1.
\end{aligned}
\tag{2}
$$

$\mathsf{E}_{F^{\xi^1}}$, $\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}$, $k = 1,\, \ldots,\, M-1$ denote the operators of mathematical expectation corresponding to F^{ξ^1}, $F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}$.

Evidently, generally, a complete information on the system

$$
\mathcal{F} = \{F^{\xi^1}(z^1), \quad F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 2, \ldots,\, M\} \tag{3}
$$

is a necessary condition to solve the above defined problem. However in applications very often one of the following cases happen: The system (3) must be replaced by its statistical estimates; the system (3) must be (for numerical difficulties) replaced by some simpler one; the actual system (3) is a little modified (e.g. by a contamination). Consequently, a stability (considered with respect to a probability measures space) and empirical estimates in the multistage problems have been (in the stochastic programming literature) investigated (see e.g. [3], [4], [9]). Approximation solution schemes (in the case of "deterministic" constraints set) have been suggested e.g. in [13] (see also [4]). We focus to a special case when the random element follows an autoregressive random sequence and the constraints sets correspond to a system of the individual probability constraints. In details we assume:

A.1 $\{\xi^k\}_{k=-\infty}^{\infty}$ follows a nonlinear autoregressive sequence

$$
\xi^k = H(\xi^{k-1}) + \varepsilon^k, \tag{4}
$$

where ξ^0, ε^k, $k = 1,\, 2,\, \ldots$ are stochastically independent (ε^k, $k = 1, \ldots$ identically distributed), $H(z)$ is a Lipschitz function on R^s (we denote the distribution function corresponding to ε^1 by the symbol F^{ε} and suppose the realization ξ^0 to be known),

A.2 there exist $g^{k+1}_i(\bar{x}^{k+1})$, $k = 1,\, 2 \ldots,\, M-1$ defined on $R^{n(k+1)}$ and $\alpha_i \in (0,\, 1)$, $i = 1,\, \ldots,\, s$, $\bar{\alpha} = (\alpha_1,\, \ldots,\, \alpha_s)$ such that

$$
\begin{aligned}
&\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) \quad (:= \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k;\, \bar{\alpha})) \quad = \\
&\bigcap_{i=1}^{s} \{x^{k+1} \in X^{k+1} :\, P_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}\{g^{k+1}_i(\bar{x}^{k+1}) \le \xi^{k+1}_i\} \ge \alpha_i\}.
\end{aligned}
\tag{5}
$$

This special case has been already investigated in [5]. In this contribution we go deeper. Our stability results will be based on the $\mathcal{L}_1$ norm and, moreover we intend to investigate (separately) the stability of the constraints sets; the last is important from applications point of view. To this end we employ the results achieved for one–stage problems. To justify this approach see e.g. [3].