The Capacity-\(C\) Torch Problem

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Abstract. The torch problem (also known as the bridge problem or the flashlight problem) is about getting a number of people across a bridge as quickly as possible under certain constraints. Although a very simply stated problem, the solution is surprisingly non-trivial. The case in which there are just four people and the capacity of the bridge is two is a well-known puzzle, widely publicised on the internet. We consider the general problem where the number of people, their individual crossing times and the capacity of the bridge are all input parameters. We present an algorithm that determines the shortest total crossing time; the number of primitive computations executed by the algorithm (i.e. the worst-case time complexity of the algorithm) is proportional to the square of the number of people.

Keywords: algorithm derivation, shortest path, dynamic programming, algorithmic problem solving.

The (capacity-\(C\)) torch problem is as follows.

\(N\) people wish to cross a bridge. It is dark, and it is necessary to use a torch when crossing the bridge, but they only have one torch between them. The bridge is narrow and at most \(C\) people can be on it at any one time. The people are numbered from 1 thru \(N\). Person \(i\) takes time \(t_i\) to cross the bridge; when a group of people cross together they must all proceed at the speed of the slowest.

Construct an algorithm that will get all \(N\) people across in the shortest time. Provide a clear justification that the algorithm does indeed find the shortest time.

The torch problem is an abstraction from a problem involving four people wishing to cross a bridge of capacity two and with specific concrete times. In this form, the problem is believed to have first appeared in 1981. Rote [3] gives a comprehensive bibliography.

The main interest in the torch problem is that what is “obvious” or “intuitive” is often wrong. For example, the “obvious” solution of letting the fastest person repeatedly accompany \(C-1\) people across the bridge is wrong. (If \(N=4\), \(C=2\) and the travel times are 1, 1, 2 and 2, this solution takes time 7 whereas the
shortest crossing time is 6.) Also, the “obvious” property that the shortest time is achieved when the number of crossings is minimised is incorrect. (If \( N = 5 \), \( C = 3 \) and the travel times are 1, 1, 4, 4 and 4, the shortest time is 8, which is achieved using 5 crossings. The shortest time using 3 crossings is 9.) It is not difficult to determine an upper bound on the crossing time, even in the general case. Nor is it difficult to provide counterexamples to incorrect solutions. The difficulty is to establish an irrefutable lower bound on the crossing time. A proper solution to the problem poses a severe test of our standards of proof.

In our solution, we assume that the people are ordered so that \( t_i < t_j \) if \( i < j \). If the given times are such that \( t_i = t_j \) for some \( i \) and \( j \), where \( i < j \), we can always consider pairs \((t_i, i)\), where \( i \) ranges over people, ordered lexicographically. Renaming the crossing “times” to be such pairs, we obtain a total ordering on times with the desired property. We also assume that \( N \) is at least \( C + 1 \). (When \( N \) is at most \( C \), it is obvious that exactly one crossing gives the optimal solution. When \( N \) is at least \( C + 1 \), more than one crossing is required.)

For brevity, some of the more straightforward proofs at the beginning of the paper. A full version of the paper, which includes the details of all proofs, is available from the author’s website.

1 Outline Strategy

An outline of our solution is as follows.

We call a sequence of crossings that gets everyone across in accordance with the rules a putative sequence. We will say that one putative sequence subsumes another putative sequence if the time taken by the first is at most the time taken for the second. Note that the subsumes relation is reflexive (every putative sequence subsumes itself) and transitive (if putative sequence \( a \) subsumes putative sequence \( b \) and putative sequence \( b \) subsumes putative sequence \( c \) then putative sequence \( a \) subsumes putative sequence \( c \)). An optimal sequence is a putative sequence that subsumes all putative sequences. A putative sequence is suboptimal if it is not optimal. The problem is to find an optimal sequence.

Recall that, when crossing the bridge, the torch must always be carried. This means that crossings, both of groups of people and of each individual person, alternate between “forward” and “return” trips, where a forward trip is a crossing in the desired direction, and a return trip is a crossing in the opposite direction.

A regular forward trip means a crossing in the desired direction made by at least two people, and a regular return trip means a trip in the opposite direction made by exactly one person. A regular sequence is a putative sequence that consists entirely of regular forward and return trips.

1 Our examples are chosen so that it is easy for the reader to discover the fastest crossing time. Of course, the examples in puzzle books are deliberately chosen to make it difficult.

2 Strictly, we also need to extend addition to pairs. Defining \((t, i) + (u, j)\) to be \((t + u, i | j)\) guarantees the appropriate algebraic structure, in particular distributivity of addition over minimum.