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Bub–Clifton Theorem

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The two fundamental 'no go' theorems for hidden variable reconstructions of the ► quantum statistics, the ► Kochen-Specker theorem [4] and ► Bell's theorem [1], can be formulated as results about the impossibility of associating a classical probability space $(X, \mathfrak{F}, P_\rho)$ with a quantum system in the state ρ, when certain constraints are placed on the probability measure P_ρ. The Bub–Clifton theorem [2, 3], by contrast, is a 'go' theorem: a positive result about the possibility of associating a classical probability space with a quantum system in a given state.

If P_ρ is required to satisfy the conditions:

(a) $P_\rho(a, b, \ldots | A, B, \ldots)$ is a classical probability measure defined for all eigenvalues $a, b, \ldots$ of the ► observables $A, B, \ldots$ in some set of observables $\mathcal{E}$.
(b) If $A, A', \ldots \in \mathcal{E}$ commute, then $P_\rho(a, a', \ldots | A, A', \ldots)$ coincides with the quantum mechanical probability assigned by ρ.

then the existence of P_ρ is equivalent to the requirement that the set of numbers:

$$\{P_\rho(a, a', \ldots | A, A', \ldots);\ A, A' \in \mathcal{E} \text{ commute}\}$$

should satisfy a finite family of inequalities (Boole's 'conditions of possible experience'), so the non-existence of P_ρ entails a violation of at least one inequality (see Pitowsky [6, 7]). If P_ρ exists, then it is a weighted average of pure states (characteristic functions onto 1-element subsets of X or 2-valued (0,1) probability measures).

The Kochen-Specker and Bell theorems can be formulated (following Pitowsky) as follows:

The Kochen-Specker Theorem. *There is a set of observables $\mathcal{E}$ such that for all ρ the classical probability measure P_ρ does not exist.*

Bell's Theorem. *There is a set of local observables $\mathcal{E}$ on $\mathcal{H} \otimes \mathcal{H}$ and a state $\rho \in \mathcal{H} \otimes \mathcal{H}$ such that the classical probability measure P_ρ does not exist.*

The Bub–Clifton is the positive result:

The Bub–Clifton Theorem. *For every pure state $\rho = |\psi\rangle\langle\psi|$ and every observable R, there is a maximal extension $\mathcal{E}$ of $\{R\}$ for which there exists a classical probability measure P_ρ. The extension $\mathcal{E}$ is unique if we require invariance with respect to automorphisms of the subspace structure of $\mathcal{H}$ (the projective geometry of $\mathcal{H}$) that preserve ρ and R.*

The pure state ρ can be expressed as a linear ▶ superposition of orthogonal 1-dimensional projection operators (▶ projection) ρ_r onto the non-null eigenspaces $\{V_r\}$ of R: $\rho = \bigvee_r \rho_r = \sum_r \rho_r$. The theorem shows that the set of observables $\mathcal{E}$ contains all the maximal observables whose spectral measures comprise:

(i) The 1-dimensional projection operators ρ_r,
(ii) The 1-dimensional projection operators onto any orthogonal basis in the orthocomplement of the subspace spanned by the projections ρ_r, i.e., the 'null space' V_{null} that is the range of the projection operator $I - \sum_r \rho_r$,

and all the non-maximal observables which are functions of these maximal observables.

Equivalently, $\mathcal{E}$ consists of all the observables whose eigenspaces are spanned by the rays defined by (i) and (ii) above.

According to the theorem, even though the set $\mathcal{E}$ contains non-commuting observables, there exists a classical probability measure P_ρ for the observables in $\mathcal{E}$, i.e., a measure space $(X, \mathfrak{F}, P_\rho)$, where the elements of the space X are the projection operators ρ_r, which are in 1-1 correspondence with the 2-valued homomorphisms—representing bivalent truth-value assignments—on the lattice of subspaces generated by the 1-dimensional projectors in (i) and (ii) above, and hence in 1-1 correspondence with the 2-valued homomorphisms on the ranges of values of the observables in $\mathcal{E}$.

Nakayama [5] has constructed a topos-theoretic extension of the theorem.

A quantum measurement interaction can be represented schematically as follows:

$$|s\rangle|r\rangle \xrightarrow{U(t)} \sum_i c_i|s_i\rangle|r_i\rangle$$

where $|s\rangle = \sum_i c_i|s_i\rangle$ is the initial state of the measured system expressed as a linear superposition of the eigenstates $|s_i\rangle$ of the measured observable S, $|r\rangle$ is the initial state of the measuring instrument with indicator or 'pointer' observable R, and $U(t)$ is the unitary transformation implementing the measurement interaction between the system and the measuring instrument that sets up a correlation between eigenvalues of S and pointer positions. (Note that for the systems we use as measuring instruments, the pointer observable R commutes with the instrument-environment interaction Hamiltonian, so the correlation between eigenvalues of S and pointer positions R induced by the system-instrument Hamiltonian is preserved under the instrument-environment interaction.) If we take the pointer observable R as 'preferred,' in the sense that it always has a definite (determinate) value, then