Fractional Vertex Arboricity of Graphs

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Abstract. The vertex arboricity $va(G)$ of a graph $G$ is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. The fractional version of vertex arboricity is introduced in this paper. We determine fractional vertex arboricity for several classes of graphs, e.g., complete multipartite graphs, cycles, integer distance graphs, prisms and Peterson graph.

Keywords: vertex arboricity; tree coloring; fractional vertex arboricity; fractional tree coloring.

1 Introduction

In this paper, we use $\mathbb{Z}$ to denote the set of all integers and $|S|$ for the cardinality of a set $S$ ($|S|=+\infty$ means that $S$ is an infinite set).

A $k$-coloring of a graph $G$ is a mapping $g$ from $V(G)$ to \{1, 2, \ldots, $k$\}. With respect to a given $k$-coloring, $V_i$ denotes the set of all vertices of $G$ colored with $i$, and $\langle V_i \rangle$ denotes the subgraph induced by $V_i$ in $G$. If $V_i$ induces a subgraph whose connected components are trees, then $g$ is called a $k$-tree coloring. The vertex arboricity of a graph $G$, denoted by $va(G)$, is the minimum integer $k$ for which $G$ has a $k$-tree coloring. In other words, the vertex arboricity $va(G)$ of $G$ is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph (i.e., a forest).

In fact, if $V_i$ is an independent set for each $i$ (1 $\leq$ $i$ $\leq$ $k$), then $g$ is called a proper $k$-coloring and the chromatic number $\chi(G)$ of a graph $G$ is the minimum integer $k$ of colors for which $G$ has a proper $k$-coloring. So the proper coloring is a special case of the tree coloring.

Kronk and Mitchem \cite{1} proved that $va(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph $G$. Chartrand etc. \cite{2} showed $va(K(p_1, p_2, \ldots, p_n)) = n - \max\{k | \sum_{i=0}^{k} p_i \leq n-k\}$ for the complete $n$-partite graph $K(p_1, p_2, \ldots, p_n)$, where $p_0=0$, $1 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

In this paper, we introduce the fractional version of vertex arboricity and to determine fractional vertex arboricity for several families of graphs. This is

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the first paper in a series of investigations on fractional vertex arboricity, its relationship with other graphic parameters.

2 Fractional Vertex Arboricity of Graphs

Let $S$ be a set of subsets of a set $V$. A covering of $V$ is a collection of elements $L_1, L_2, \ldots, L_j$ of $S$ such that $V \subseteq L_1 \cup \cdots \cup L_j$.

For any graph $G$, let $\mathcal{F}(G)$ be the set of all subsets of $V(G)$ that induce forests of $G$.

We now define the fractional vertex arboricity $va_f(G)$ of a graph $G$ as follows.

**Definition 1.** A fractional tree coloring of a graph $G$ is a mapping $g$ from $\mathcal{F}(G)$ to the interval $[0, 1]$ such that

$$\sum_{L \text{ contains } x} g(L) \geq 1, \text{ for any } x \in V(G).$$

The weight of a fractional tree coloring is the sum of its values, and the fractional vertex arboricity of a graph $G$ is the minimum possible weight of a fractional coloring, that is,

$$va_f(G) = \min \left\{ \sum_{L \in \mathcal{F}(G)} g(L) \mid g \text{ is a fractional tree coloring of } G \right\}.$$ 

Clearly, we have $va_f(H) \leq va_f(G)$ for any subgraph $H$ of $G$.

If we restrict the range of a mapping $g$ to $\{0, 1\}$ instead of $[0, 1]$, then $va_f(G)$ is the usual vertex arboricity, $va(G)$.

If $g$ is a $va(G)$-tree coloring of $G$ and $V_i = \{v \mid v \in V(G), g(v) = i\}$ $(1 \leq i \leq va(G))$, then we can define a mapping $h : \mathcal{F}(G) \longrightarrow [0, 1]$ by

$$h(L) = \begin{cases} 
1, & \text{for } L = V_i, 1 \leq i \leq va(G), \\
0, & \text{otherwise}. 
\end{cases}$$

such that $h$ is a fractional tree coloring of $G$ which has the weight $va(G)$. Therefore, it follows immediately that $va_f(G) \leq va(G)$.

Conversely, if $G$ has a $(0, 1)$-valued fractional tree coloring $g$ of weight $k$. Then the support of $g$ consists of $k$ forests $V_1, V_2, \ldots, V_k$ whose union is $V(G)$. If we color any vertex $v$ with the smallest $i$ such that $v \in V_i$, then we have a $k$-tree coloring of $G$. Thus the vertex arboricity of $G$ is the minimum weight of a $(0, 1)$-valued fractional tree coloring.

**Remark 1.** Vertex arboricity of a finite graph $G$ can be seen as an optimal solution of an integer programming and its fractional version can be viewed as an optimal solution of its relaxed problem, i.e., a linear programming problem.