14 The Quadratic Quadrilateral Element

14.1 Basic Equations

The quadratic quadrilateral element is a two-dimensional finite element with both local and global coordinates. It is characterized by quadratic shape functions in each of the $x$ and $y$ directions. This element can be used for plane stress or plane strain problems in elasticity. This is the second isoparametric element we deal with in this book. The quadratic quadrilateral element has modulus of elasticity $E$, Poisson’s ratio $\nu$, and thickness $t$. Each quadratic quadrilateral element has eight nodes with two in-plane degrees of freedom at each node as shown in Fig. 14.1. The global coordinates of the eight nodes are denoted by $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$, $(x_5, y_5)$, $(x_6, y_6)$, $(x_7, y_7)$, and $(x_8, y_8)$. The order of the nodes for each element is important – they should be listed in a counterclockwise direction starting from the corner nodes followed by the midside nodes. The area of each element should be positive – you can actually check this by using the MATLAB function `QuadraticQuadElementArea` which is written specifically for this purpose. The element is mapped to a rectangle through the use of the natural coordinates $\xi$ and $\eta$ as shown in Fig. 14.2. In this case the element stiffness matrix is not written explicitly but calculated through symbolic integration with the aid of the MATLAB Symbolic Math Toolbox. The eight shape functions for this element are listed explicitly as follows in terms of the natural coordinates $\xi$ and $\eta$ (see [1]).

![Fig. 14.1. The Quadratic Quadrilateral Element](image)
The Quadratic Quadrilateral Element

\[ N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1) \]
\[ N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(\xi - \eta - 1) \]
\[ N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(\xi + \eta - 1) \]
\[ N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1) \]
\[ N_5 = \frac{1}{2}(1 - \eta)(1 + \xi)(1 - \xi) \]
\[ N_6 = \frac{1}{2}(1 + \xi)(1 + \eta)(1 - \eta) \]
\[ N_7 = \frac{1}{2}(1 + \eta)(1 + \xi)(1 - \xi) \]
\[ N_8 = \frac{1}{2}(1 - \xi)(1 + \eta)(1 - \eta) \] (14.1)

The Jacobian matrix for this element is given by

\[
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\] (14.2)

where \( x \) and \( y \) are given by

\[
x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 + N_5 x_5 + N_6 x_6 + N_7 x_7 + N_8 x_8
\]
\[
y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 + N_5 y_5 + N_6 y_6 + N_7 y_7 + N_8 y_8 \] (14.3)

The \([B]\) matrix is given as follows for this element:

\[
[B] = [D'][N] \] (14.4)