Relational Galois Connections

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Abstract. Galois connections can be defined for lattices and for ordered sets. We discuss a rather wide generalisation, which was introduced by Weiqun Xia and has been reinvented under different names: Relational Galois connections between relations. It turns out that the generalised notion is of importance for the original one and can be utilised, e.g., for computing Galois connections.

The present paper may be understood as an attempt to bring together ideas of Wille [15], Xia [16], Domenach and Leclerc [3], and others and to suggest a unifying language.

1 Galois Connections Between Relations

It is usual to define a **Galois connection** between two complete lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ as a pair (φ, ψ) of mappings

$$\varphi : L_1 \to L_2, \quad \psi : L_2 \to L_1$$

satisfying for all $x \in L_1$ and all $y \in L_2$

$$x \leq_1 \psi(y) \iff y \leq_2 \varphi(x).$$

It is well known that if this condition is satisfied, both mappings φ and ψ are order reversing and that their compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are closure operators on L_2 and L_1, respectively, with dually isomorphic lattices of closed sets (see, e.g., [4]).

Not so obvious is how to *construct* Galois connections for given lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$. We shall address this question later. For the case that both lattices are power set lattices, the answer was given by Birkhoff [2]: Let G and M be sets, let $I \subseteq G \times M$ be some relation. Define

$$A^I := \{m \in M \mid g\ I\ m \text{ for all } g \in A\} \qquad \text{if } A \subseteq G$$

and

$$B^I := \{g \in G \mid g\ I\ m \text{ for all } m \in B\} \qquad \text{if } B \subseteq M.$$

Then

$$\varphi(X) := X^I \quad \text{for } X \subseteq G \qquad \text{and} \qquad \psi(Y) := Y^I \quad \text{for } Y \subseteq M$$

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defines a Galois connection between the power set lattice of G and the power set lattice of M. The set

$$\mathfrak{B}(G, M, I) := \{(A, B) \mid A \subseteq G, B \subseteq M, A^I = B, A = B^I\},$$

ordered by

$$(A_1, B_1) \leq (A_2, B_2) :\iff A_1 \subseteq A_2 \iff B_2 \subseteq B_1$$

is a complete lattice, called the **concept lattice**[1]. The sets of the form A^I, $A \subseteq G$, are called the **intents** of (G, M, I), and those of the form B^I, $B \subseteq M$ are the **extents**. These are the closed sets of the two closure operators.

The notion of a Galois connection can be generalised to ordered sets and, even further, to arbitrary binary relations $I \subseteq G \times M$, $J \subseteq H \times N$, as in the next definition:

Definition 1. *A* **Galois connection**[2] *between* (G, M, I) *and* (H, N, J) *is a pair* (φ, ψ) *of mappings*

$$\varphi : G \to N, \quad \psi : H \to M$$

satisfying

$$g \; I \; \psi(h) \iff h \; J \; \varphi(g).$$

This definition is symmetric: if (φ, ψ) is a Galois connection between (G, M, I) and (H, N, J), then (ψ, φ) is a Galois connection between (H, N, J) and (G, M, I). This corresponds to the original, *contravariant* definition of Galois connections. Some authors consider also the *covariant* version, which allows for composition of Galois connections. This is achieved when (H, N, J) is replaced by the *dual context* (N, H, J^{-1}). These mappings are closely related to *infomorphisms* and to *Chu morphisms*, see Section 6 for more.

One might argue that this definition deviates from the original one for lattices or ordered sets. But it is only a natural generalisation. For two ordered sets $(P, \leq_1)$ and $(Q, \leq_2)$ the condition that (φ, ψ) is a Galois connection between $(P, P, \leq_1)$ and $(Q, Q, \leq_2)$ is

$$x \leq_1 \psi(y) \iff y \leq_2 \varphi(x),$$

as usual.

We may generalise even further, replacing the pair of mappings by a pair of relations $\Phi \subseteq G \times N$ and $\Psi \subseteq H \times M$. The natural condition then is that

$$g \; I \; h^{\Psi} \iff h \; J \; g^{\Phi}$$

holds for all $g \in G$ and all $h \in H$. We call this the (relational) **Galois condition**. However, this condition by itself turns out to be not strong enough. We therefore define

[1] Our notation is that of Formal Concept Analysis [5], where (G, M, I) is called a **formal context**. Other authors use names like **classification** [1], **Chu-space** [11], etc.

[2] Called **context–Galois connection** by Xia [16].