Some Notes on Pseudo-closed Sets*

Sebastian Rudolph

Institute AIFB
University of Karlsruhe (TH)
Germany
rudolph@aifb.uni-karlsruhe.de

Abstract. Pseudo-intents (also called pseudo-closed sets) of formal contexts have gained interest in recent years, since this notion is helpful for finding minimal representations of implicational theories. In particular, there are some open problems regarding complexity. In our paper, we compile some results about pseudo-intents which contribute to the understanding of this notion and help in designing optimized algorithms. We provide a characterization of pseudo-intents based on the notion of a formal context's incrementors. The latter are essentially non-closed sets which – when added to a closure system – do not enforce the presence of other new attribute sets. In particular, the provided definition is non recursive. Moreover we show that this notion coincides with the notion of a quasi-closed set that is not closed, which enables to reuse existing results and to formulate an algorithm that checks for pseudo-closedness. Later on, we provide an approach for further optimizing those algorithms based on a result which correlates the set of pseudo-intents of a formal context with the pseudo-intents of this context's reduced version.

1 Introduction

Pseudo-intents are of significant interest in formal concept analysis. One central result ([5]) states, that the implication set $\{P \to P^{II} \mid P \text{ pseudo-intent of } \mathbb{K}\}$ (called *stem base*) constitutes a so-called *implicational base*, i.e., a minimal set of implications generating the implicational theory of the formal context $\mathbb{K}$. In this regard it is also important to note that for an arbitrary implication, checking whether it is semantically entailed by a set of implications can be decided in linear time ([2, 8]). Thus, pseudo-intents become relevant for problems related to small (yet quick to query) representation of implicative knowledge.

The complexity of determining for a given context $\mathbb{K} = (G, M, I)$ and attribute set $A \subseteq M$, whether A is a pseudo-intent (or: pseudo-closed) with respect to $\mathbb{K}$ is still an open problem (see [9]). The prevailing assumption seems to be that the problem's complexity is rather high (at least beyond polynomial time). Partial results ([7, 6]) show that it is in coNP.

In our paper, we compile some results about pseudo-intents and provide optimized algorithms for checking for pseudo-closedness.

In detail, we will proceed as follows: In Section 2, we recall the fundamental definitions and propositions of FCA needed to specify and deal with the topic. Section 3

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provides and verifies an algorithm which allows to convert an arbitrary set of implications into a stem base. Section 4 introduces the notion of *incrementor* and shows how it can be used to provide a non-recursive characterization of pseudo-intents. In the end, this notion shows to have a direct correspondence to that of a *quasi intent* introduced in [3]. Resulting from these preceding considerations, Section 5 presents an algorithm which checks for pseudo-closedness. Section 6 shows how pseudo-closedness can be checked even by examining the reduced version of the considered context and provides a corresponding algorithm. Finally, Section 7 concludes and outlines possible directions for further research.

2 Preliminaries

In this section, we will introduce the notions from formal concept analysis necessary for our work.

First of all, note that we use the notation "$\subset$" to indicate the *strict* subset, i.e. $A \subset B$ means $A \subseteq B$ and $A \neq B$.

Deviating from the usual line of presentation, we will introduce implications and pseudo-closed sets just on the basis of closure operators. This allows to talk about those notions independently from concrete formal concepts and facilitates the presentation of some results in the sequel. However, note that this is not a proper generalization, since every closure operator can be represented by the $(.)^{II}$-operator of an appropriately chosen formal context (e.g. the context $(\{A \mid \varphi(A) = A\}, M, \ni)$). Thus, the cited definitions and results – although defined on basis of a formal context – carry over to our way of introducing those notions.

The following considerations are based on an arbitrary set M. We will first define the fundamental notion of a closure operator M. Roughly spoken, applying such an operator to a set can be understood as a minimal extension of that set in order to fulfill certain properties.

Definition 1. *Let M be an arbitrary set. A function $\varphi : \mathcal{P}(M) \to \mathcal{P}(M)$ (where $\mathcal{P}(M)$ denotes the powerset of M) will be called*

- EXTENSIVE, *if $A \subseteq \varphi(A)$ for all $A \subseteq M$,*
- MONOTONE, *if from $A \subseteq B$ follows $\varphi(A) \subseteq \varphi(B)$ for all $A, B \subseteq M$, and*
- IDEMPOTENT, *if $\varphi(\varphi(A)) = \varphi(A)$ for all $A \subseteq M$.*

If φ is extensive, monotone, and idempotent, we will call it a CLOSURE OPERATOR. *In this case, we will additionally call*

- *$\varphi(A)$ the* CLOSURE *of A,*
- *A* CLOSED *(or φ-*CLOSED*), if $A = \varphi(A)$.*

The family of all closed sets is also called CLOSURE SYSTEM. Furthermore, any closure system constitutes a *lattice* with set inclusion as the respective order relation.

In the sequel, we show, in which way closure operators are closely related to implications.