On Multi-adjoint Concept Lattices: Definition and Representation Theorem

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Abstract. Several fuzzifications of formal concept analysis have been proposed to deal with uncertainty or incomplete information. In this paper, we focus on the new paradigm of multi-adjoint concept lattices which embeds different fuzzy extensions of concept lattices, our main result being the representation theorem of this paradigm. As a consequence of this theorem, the representation theorems of the other paradigms can be proved more directly. Moreover, the multi-adjoint paradigm enriches the language providing greater flexibility to the user.

Keywords: concept lattices, multi-adjoint lattices, Galois connection, implication triples.

1 Introduction

The study of reasoning methods under uncertainty, imprecise data or incomplete information has shown to be an important topic in the recent years. Most of the current research areas are receiving this message and it is frequent to see *fuzzified* versions of several well-known standard structures. In this paper, we focus on the area of formal concept analysis and, specifically, on the generalization of the classical definition of concept lattice to the fuzzy case.

A number of different approaches have been proposed to generalize the classical concept lattices given by Ganter and Wille [10] allowing some uncertainty in data, a recent survey and comparison of approaches to fuzzy concept lattices is presented in [6].

One of these approaches was proposed by Burusco and Fuentes-González [7] where fuzzy concept lattices were first presented, and later further developed by Pollandt [23] and Bělohlávek [2] who use complete residuated lattices as structures for the truth degrees. For the latter approach, the main theorem was proved in two ways: firstly, by reduction to the crisp version of main theorem (this was proved independently in [23] and, more generally, via representation of fuzzy Galois connections in [4]); secondly, working directly in a fuzzy setting in [3].

Bělohlávek, in [5], later extended this to the case when a fuzzy partial order is considered on a fuzzy concept lattice instead of on an ordinary partial order.

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Georgescu and Popescu extended this framework to non-commutative logic and similarity in a series of papers [11, 12, 13, 14]; in a different direction, it was also extended in an asymmetric way, although only for the case of classical equality ($L = \{0, 1\}$), by Krajči, which introduced the so-called generalized concept lattices in [17, 18].

In the context of general logical frameworks, a recent approach so-called multi-adjoint has been recently introduced and is receiving considerable attention [16, 21]. The multi-adjoint framework was originated as a generalization of several non-classical logic programming frameworks, its semantic structure is the multi-adjoint lattice, in which a lattice is considered together with several conjunctors and implications making up adjoint pairs [20].

In [22], with the idea of providing a general framework in which the different approaches stated above could be conveniently accommodated, the authors considered a general non-commutative environment; this naturally leads to the consideration of adjoint triples, also called pre-implication triples [1] or bi-residuated structures [19] as the main building blocks of our multi-adjoint concept lattices.

The aim of the paper is to construct so-called multi-adjoint concept lattices in order to generalise different fuzzy extensions of concept lattices. The main result is a representation theorem which characterises those complete lattices which are isomorphic to multi-adjoint concept lattices. The notion of a multi-adjoint concept lattice is demonstrated by a detailed example.

The plan of this paper is the following: in Section 2 we recall the basics about Galois connection and the notion of multi-adjoint concept lattice is introduced, in Section 3 contains the proof of the representation theorem; in Section 4 an example of the multi-adjoint framework is presented; the paper ends with some conclusions and prospects for future work.

2 Multi-adjoint Concept Lattice

A basic notion in formal concept analysis is that of *Galois connection*, we start this section recalling a result which proves that each Galois connection has an associated complete lattice, called *Galois lattice* or *concept lattice*.

Definition 1. *Let* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *be posets, and let* ${}^{\downarrow}\colon P_1 \to P_2$ *and* ${}^{\uparrow}\colon P_2 \to P_1$ *be mappings, the pair* $({}^{\uparrow}, {}^{\downarrow})$ *forms a* Galois connection *between* P_1 *and* P_2 *if:*

1. ${}^{\uparrow}$ *and* ${}^{\downarrow}$ *are order-reversing.*
2. $x \leq_1 x^{\downarrow\uparrow}$ *for all* $x \in P_1$*.*
3. $y \leq_2 y^{\uparrow\downarrow}$ *for all* $y \in P_2$*.*

If P_1 and P_2 are complete lattices then the following theorem can be established, see [9], which will be used in order to prove that our construction of multi-adjoint concept lattices actually leads to a complete lattice.