Base Points, Non-unit Implications, and Convex Geometries

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Abstract. We study the "non-unit implications" of a formal context and investigate the closure system induced by these implications. It turns out that this closure system is the largest closure system on the same base set containing the given one as a complete sublattice. This was studied by other authors with special emphasis on semidistributivity and convex geometries. We present some of their results in FCA language.

The complete lattice refinements of a closure system form an interval within the lattice of all closure systems. We describe the reduced context for this interval.

For better compatibility with the literature, we dualize and consider implications between objects, not attributes.

1 Complete Lattice Refinements of Closure Systems

We study closure systems on a set G. For simplicity, and w.l.o.g., we assume each such closure system to be given as the system $\mathrm{Ext}(\mathbb{K})$ of extents of some formal context $\mathbb{K} := (G, M, I)$. The corresponding closure operator will be written as $X \mapsto X^{II}$, or simply as $X \mapsto X''$.

A *refinement* of a closure system is another closure system on the same base set, containing the first as a subset. Such refinements can be obtained as the systems of extents of appositions $\mathbb{K} \,|\, \mathbb{L}$, where $\mathbb{L}$ is an arbitrary formal context with object set G, cf. [GW99, Definition 30]. The lattice $(\mathrm{Ext}(\mathbb{K}), \subseteq)$ is always a $\bigwedge$-subsemilattice of $(\mathrm{Ext}(\mathbb{K} \,|\, \mathbb{L}), \subseteq)$, but usually not a sublattice, because joins may not be preserved. In the exceptional case that $(\mathrm{Ext}(\mathbb{K}), \subseteq)$ is a (complete) sublattice of $(\mathrm{Ext}(\mathbb{K} \,|\, \mathbb{L}), \subseteq)$, we speak of a *(complete) lattice refinement.*

In the finite case, the two notions (of lattice refinement and complete lattice refinement) conincide iff $\emptyset$ is closed. Lattice refinements of finite closure systems have been studied by Adaricheva and Nation [AN03], [N04].

Lemma 1. *Let* $\mathbb{K} := (G, M, I)$ *and* $\mathbb{L} := (G, N, J)$*. Then* $\mathrm{Ext}(\mathbb{K} \,|\, \mathbb{L})$ *is a complete lattice refinement of* $\mathrm{Ext}(\mathbb{K})$ *if and only if for each* $n \in N$ *the set*

$$\Theta(n) := \{g \in G \mid g^{II} \subseteq n^J\}$$

is an extent of $\mathbb{K}$.

S.O. Kuznetsov and S. Schmidt (Eds.): ICFCA 2007, LNAI 4390, pp. 210–220, 2007.

Proof. Clearly $\Theta(n)^{II}$ is an extent of $\mathbb{K}$. More precisely, it is the supremum of all $g^{II} \subseteq n^J$ in $\mathrm{Ext}(\mathbb{K})$. In $\mathrm{Ext}(\mathbb{K} \mid \mathbb{L})$, n^J is an upper bound of $\{g \in G \mid g^{II} \subseteq n^J\}$, and in order to preserve joins it is necessary that $\Theta(n)^{II} \subseteq n^J$. Then we have $\Theta(n)^{II} = \Theta(n)$.

Now consider an arbitrary family $\mathcal{F} \subseteq \mathrm{Ext}(\mathbb{K})$ and let $U := \bigcup \mathcal{F}$. Then U^{II} is the join of $\mathcal{F}$ in $\mathrm{Ext}(\mathbb{K})$, and the join in $\mathrm{Ext}(\mathbb{K} \mid \mathbb{L})$ is different iff there exists some attribute $n \in N$ such that n^J contains U but not U^{II}. Since U is a union of extents, we have that

$$g \in U \Rightarrow g^{II} \subseteq U,$$

and thus $U \subseteq \Theta(n) \subseteq n^J$. If this is an extent of $\mathbb{K}$, it is an upper bound of $\mathcal{F}$ and thus contains U^{II}, a contradiction. □

It is quite evident (and follows from Theorem 1) that the intersection of complete lattice refinements of $\mathrm{Ext}(\mathbb{K})$ again yields a complete lattice refinement. The lattice refinements of $\mathrm{Ext}(\mathbb{K})$, ordered by inclusion, therefore form a complete lattice. (However, lattice refinements of $\mathrm{Ext}(\mathbb{K})$ need not be lattice refinements of each other, see Example 1). We shall discuss some properties of this lattice in Section 2.

Let us call a closure system on G *elementary*, if each singleton set $\{g\}$ is closed ($g \in G$). If $|G| > 1$, then a closure system on G can only be elementary if the empty set $\emptyset$ is closed. The following proposition states that a closure system has a proper complete lattice refinement, i.e. $\mathrm{Ext}(\mathbb{K}) \subsetneq \mathrm{Ext}(\mathbb{K} \mid \mathbb{L})$, iff $\mathrm{Ext}(\mathbb{K})$ is not elementary.

Proposition 1. *Each closure system containing $\emptyset$ has an elementary complete lattice refinement. No elementary closure system admits a proper complete lattice refinement.*

Proof. The condition of Lemma 1 is automatically fulfilled when the empty set is closed and $|n^J| = 1$. Therefore, $\mathrm{Ext}(\mathbb{K} \mid (G, G, =))$ always is an elementary complete lattice refinement if $\emptyset$ is closed.

If the closure system $\mathrm{Ext}(\mathbb{K})$ is elementary then $g = g^{II}$ and thus $\{g \in G \mid g^{II} \subseteq n^J\} = n^J$. The condition of Lemma 1 requires in this case that all attribute extents of $\mathbb{L}$ are already extents of $\mathbb{K}$. □

Remark 1. Proposition 1 does not hold for lattice refinements in general (without the prefix "complete"). As an example, consider the closure system consisting of the set of natural numbers $\mathbb{N}$ and of all its finite subsets. It is elementary, and the closure system of all subsets of $\mathbb{N}$ is a proper lattice refinement.

Example 1. Example 1 shows, on the example of $\underline{N}_5$, an elementary complete lattice refinement as given by Proposition 1 (middle, $\underline{\mathfrak{B}}(\widetilde{\mathbb{K}})$). The diagram on the right, $\underline{\mathfrak{B}}(\widehat{\mathbb{K}})$ shows another elementary complete lattice refinement. Note that $\mathrm{Ext}(\widehat{\mathbb{K}})$ refines $\mathrm{Ext}(\widetilde{\mathbb{K}})$, but is no lattice refinement. In fact, according to Proposition 1, $\mathrm{Ext}(\widetilde{\mathbb{K}})$ cannot have a proper complete lattice refinement.