# Lattices of Relatively Axiomatizable Classes*

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**Abstract.** In the paper we study lattices of axiomatizable classes and relatively axiomatizable classes. This study is based on Formal Concept Analysis [4,5]. The notion of a relatively axiomatizable class is a generalization of such concepts as variety, quasivariety, $\forall$-axiomatizable class, $\exists$-axiomatizable class, $\Pi_n^0$-axiomatizable class, $\Sigma_n^0$-axiomatizable class and so on. Relatively axiomatizable classes were studied in [10]. It is proved in the paper that any finite lattice may be represented as the lattice of all relatively axiomatizable subclasses of the class of all models of a one-element signature with respect to some set of sentences. Also we prove that any finite or countable complete lattice is isomorphic to the lattice of all relatively axiomatizable subclasses of some class of models with respect to a proper set of sentences.

**Keywords:** lattice, axiomatizable class, relatively axiomatizable class.

## 1 Introduction

The main goal of the present paper is to investigate the structure of various lattices of relatively axiomatizable classes.

We consider classes of algebraic systems of finite or countable signature $\sigma$. The sets of formulas and sentences of this signature are countable. Therefore in this case the lattices of axiomatizable classes have power which is less or equal to the continuum.

We show that any finite lattice may be represented as the lattice of all relatively axiomatizable subclasses of the class of all models of the signature consisting of one binary predicate symbol with respect to some subset of the set of all sentences over the given signature. Also we prove that any finite or countable complete lattice is isomorphic to the lattice of all relatively axiomatizable subclasses of some class of models of the given signature with respect to a proper set of sentences over the given signature.

## 2 Preliminaries in Model Theory

The aim of this section is to introduce concepts, definitions and facts on the model theory which are necessary for understanding of the proofs below. An expert in the model theory may skip this section.

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### 2.1 Semantics of First Order Predicate Logic

First we give some basic definitions of the first order predicate logic.

**Definition 1.** ***An algebraic system (a model)*** *is a tuple*

$$\mathfrak{A} =< A; P_1, \ldots, P_n, f_1, \ldots, f_m, c_1, \ldots, c_k >,$$

*where the set $A$ is called universe, $P_1, \ldots, P_n$ are predicates defined on the set $A$, $f_1, \ldots, f_m$ are functions defined on the set $A$ and $c_1, \ldots, c_k$ are constants, i.e. names of some (distinguished) elements of the set $A$. Usually the universe of the algebraic system $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$, i.e. $|\mathfrak{A}| := A$.*

*The set $\sigma =< P_1, \ldots, P_n, f_1, \ldots, f_m, c_1, \ldots, c_k >$ is called signature of the algebraic system $\mathfrak{A}$.*

**Definition 2.** *Consider a signature $\sigma =< P_1, \ldots, P_n, f_1, \ldots, f_m, c_1, \ldots, c_k >$. We give a definition of a* ***term*** *of the signature $\sigma$ by induction:*

1. *The constants $c_1, ..., c_k \in \sigma$ are terms and the variables $x_1, x_2, ...$ are terms.*
2. *If $t_1, \ldots, t_n$ are terms, $f \in \sigma$ and $f$ is a symbol of function then $f(t_1, \ldots, t_n)$ is a term.*

**Definition 3.** *We give also an inductive definition of a* ***formula*** *of the signature $\sigma$:*

1. *If $t_1$ and $t_2$ are terms then $t_1 = t_2$ is a formula; if $t_1, \ldots, t_n$ are terms and $P^n \in \sigma$ is a predicate symbol then $P(t_1, \ldots, t_n)$ is a formula.*
2. *If $\phi$, $\psi$ are formulas then $(\phi \vee \psi)$, $(\phi \& \psi)$, $(\phi \to \psi)$, $\neg\phi$, $\forall x\ \phi$ and $\exists x\ \phi$ are formulas.*

Recall that an occurrence of a variable in a formula is called **free** if it does not belong to the scope of a quantifier over this variable. A variable which has at least one free occurrence in a formula is called **free variable of the formula**. Denote by $FV(\varphi)$ the set of all free variables of a formula $\varphi$. A formula having no free variables is called **sentence**.

**Definition 4.** *For a signature $\sigma$ we denote:*
$T(\sigma) := \{t \mid t\ is\ a\ term\ of\ the\ signature\ \sigma\}$,
$F(\sigma) := \{\varphi \mid \varphi\ is\ a\ formula\ of\ the\ signature\ \sigma\}$,
$S(\sigma) := \{\varphi \mid \varphi\ is\ a\ sentence\ of\ the\ signature\ \sigma\}$ *and*
$K(\sigma) := \{\mathfrak{A} \mid \mathfrak{A}\ \ is\ a\ model\ of\ the\ signature\ \sigma\}$.

**Remark 1.** *If $\sigma_1 \subseteq \sigma_2$ then*

1. $T(\sigma_1) \subseteq T(\sigma_2)$,
2. $F(\sigma_1) \subseteq F(\sigma_2)$,
3. $S(\sigma_1) \subseteq S(\sigma_2)$,
4. *if $\sigma_1 \neq \sigma_2$, then $K(\sigma_1) \cap K(\sigma_2) = \emptyset$.*