On the MacNeille Completion of Weakly Dicomplemented Lattices

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Abstract. The MacNeille completion of a poset $(P, \leq)$ is the smallest (up to isomorphism) complete poset containing $(P, \leq)$ that preserves existing joins and existing meets. It is wellknown that the MacNeille completion of a Boolean algebra is a Boolean algebra. It is also wellknown that the MacNeille completion of a distributive lattice is not always a distributive lattice (see [Fu44]). The MacNeille completion even seems to destroy many properties of the initial lattice (see [Ha93]). Weakly dicomplemented lattices are bounded lattices equipped with two unary operations satisfying the equations (1) to (3') of Theorem 3. They generalise Boolean algebras (see [Kw04]). The main result of this contribution states that *under chain conditions* the MacNeille completion of a weakly dicomplemented lattice is a weakly dicomplemented lattice. The needed definitions are given in subsections 1.2 and 1.3.

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1 Introduction

1.1 Motivation

Concept algebras are concept lattices enriched by a weak negation and a weak opposition. They should play for Boolean Concept Logic the rôle played by the powerset algebras for Classical Propositional Logic. The class of weakly dicomplemented lattices is a variety defined by some equations valid in all concept algebras. One important and still open problem in this topic is whether every weakly dicomplemented lattice can be embedded into a concept algebra of a suitable context (called concrete embedding problem in [Kw04, Section 1.4]).

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A promising step is the prime ideal theorem. Using this result on a weakly dicomplemented lattice $(L, \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1)$, a canonical context $\mathbb{K}_{\triangledown}^{\triangle}(L)$ has been constructed and a bounded lattice embedding $\varphi : L \to \mathfrak{B}(\mathbb{K}_{\triangledown}^{\triangle}(L))$ exhibited (see Subsection 1.4), that satisfies

$$\varphi(x^{\triangledown}) \leq \varphi(x)^{\triangledown} \leq \varphi(x)^{\triangle} \leq \varphi(x^{\triangle})$$

where in $\mathcal{A}(\mathbb{K}_{\triangledown}^{\triangle}(L))$ the weak negation and weak opposition are also denoted by $^{\triangle}$ and $^{\triangledown}$ respectively. So the following question arises: does it make any difference if L is assumed to be a complete lattice? To answer this question we first examine the MacNeille completion $\tilde{L}$ of L, on which we extend the operations $^{\triangle}$ and $^{\triangledown}$. Of course $\tilde{L}$ embeds into $\mathfrak{B}(\mathbb{K}_{\triangledown}^{\triangle}(L)$. Is this a weakly dicomplemented lattice embedding of L into $\mathcal{A}(\mathbb{K}_{\triangledown}^{\triangle}(L))$? Our aim is to prove that the MacNeille completion $\tilde{L}$ of a weakly dicomplemented lattice L is a weakly dicomplemented lattice and that L embeds into $\mathcal{A}(\mathbb{K}_{\triangledown}^{\triangle}(L)$ iff $\tilde{L}$ embeds into $\mathcal{A}(\mathbb{K}_{\triangledown}^{\triangle}(L)$. Section 2 presents preliminary results for the first claim. The second claim is still not proved. Before that we recall some basic notions of Formal Concept Analysis in Subsection 1.2 and introduce weakly dicomplemented lattices in Subsection 1.3. The proofs of stated results can be found in [GW99] or [Kw04].

1.2 Formal Concept Analysis

Formal Concept Analysis is a mathematical field that aims to support human thinking. It has been introduced by Rudolf Wille in the early 80ies, and is based on the theory of lattices and ordered sets. It is started by formalizing the notions of "concept" and "concept hierarchy". The notion of *concept* is rather philosophical. A concept is considered to be determined by its extent and its intent. The extent consists of all entities belonging to the concept and the intent is the set of all common properties shared by all objects of the concept. The hierarchy on concept states that "a concept is more general if it contains more entities". For this purpose the following notions were adopted.

Definition 1. *A* **formal context** *is a triple* (G, M, I) *of sets such that* $I \subseteq G \times M$. *The members of* G *are called* **objects** *and those of* M **attributes**. *If* $(g, m) \in I$ *the object* g *is said to have* m *as an attribute. For* $A \subseteq G$ *and* $B \subseteq M$, *the* **derivation operation** $'$ *is defined by*

$$A' := \{m \in M \mid \forall g \in A \quad gIm\} \quad \textit{and} \quad B' := \{g \in G \mid \forall m \in B \quad gIm\}.$$

A **formal concept** *of* (G, M, I) *is a pair* (A, B) *with* $A \subseteq G$ *and* $B \subseteq M$ *such that* $A' = B$ *and* $B' = A$. *We call* A *the* **extent** *and* B *the* **intent** *of the concept* (A, B). *The set of all formal concepts of* (G, M, I) *is denoted by* $\mathfrak{B}(G, M, I)$. *For concepts* (A, B) *and* (C, D), *we call* (A, B) *a* **subconcept** *of* (C, D) *provided that* $A \subseteq C$ *(which is equivalent to* $D \subseteq B$*). In this case,* (C, D) *is a* **superconcept** *of* (A, B) *and we write* $(A, B) \leq (C, D)$.

The pair $(',')$ forms a Galois connection between the powersets of G and that of M. The basic theorem on concept lattices says that: