Polynomial Embeddings and Representations

Tim Becker

Institute for Medical Biometry, Informatics, and Epideomology, Germany

1 Introduction

The following paper activates polynomial methods for data analysis. We want to embed a given formal context into a polynomial context $(K^n, K[x_1, \ldots, x_n], \perp)$ in such a way that implications can be computed in the polynomial context, using the algebraic structure. The basic ideas of formal concept analysis that are needed here are sketched in [TB1].

The idea to describe given data algebraically has been investigated, for instance, in [V] where **linear embeddings** into a context of the form $(V, V^*, \perp)$ are treated. Here V is a finite-dimensional vector-space, V^* is its dual space and $a \perp \varphi$ holds if and only if $\varphi(a) = 0$.

One finds that polynomial solutions always exist, whereas empirical data in general does not allow for linear solutions. But nevertheless, polynomial solutions are not always satifactory because it is difficult to limit the (total) degree of the polynomials involved, and polynomial descriptions of a high (total) degree are not very useful for applications.

Unfortunately, if we work over an algebraically closed field, there is a large class of formal contexts which can only be embedded into a polynomial context in one variable. A problem of future research therefore could be to find suitable underlying fields or rings or even universal algebras to work with.

At the end of this paper we briefly touch polynomial representations, a related topic. As before, polynomial solutions always exist, but it is difficult to limit their total degree.

2 Polynomial Embeddings

We want to embed a given formal context $\mathbb{K}_1 := (G, M, I)$ into a context $\mathbb{K}_2$ in such a way that the closure B^{II} of a subset $B \subseteq M$ can be computed by intersecting the closure of the image of B with the image of M in $\mathbb{K}_2$. An embedding of this kind will be called an "intent-preserving embedding".

The idea is to gain a new description of the formal concepts of $\mathbb{K}_1$ and a new way to compute them, for instance, when $\mathbb{K}_2$ carries an algebraic structure.

Vogt, [V], has investigated "linear embeddings" into the context $(V, V^*, \perp)$, where V is a vector space, V^* is its dual space, and where $a \perp \varphi$ holds if and only if $\varphi(a) = 0$. He has shown that not every formal context is linearly embeddable.

So we will try to find embeddings into a polynomial context

$$(K^n, K[x_1, \ldots, x_n], \perp),$$

S.O. Kuznetsov and S. Schmidt (Eds.): ICFCA 2007, LNAI 4390, pp. 281–302, 2007.

where K is a field, $K[x_1, \ldots, x_n]$ is the polynomial ring in n variables over K and where $a \perp f$ holds if and only if $f(a) = 0$.

Definition 1. *A* **quasi-embedding** *of a formal context $\mathbb{K}_1 := (H, N, J)$ into a formal context $\mathbb{K}_2 := (G, M, I)$ is a pair (α, β), where α maps from H to G, β maps from N to M and where hIn holds if and only if $\alpha(h)J\beta(n)$ holds for all $h \in H$ and all $n \in N$.*

An **intent-preserving quasi-embedding** *of $\mathbb{K}_1$ into $\mathbb{K}_2$ is a quasi-embedding which satisfies for all $D \subseteq N$ the extra condition*

$$(1) \qquad \beta(D^{JJ}) = (\beta(D))^{II} \cap \beta(N).$$

A **linear embedding** *is an intent-preserving quasi-embedding into a linear context $(V, V^*, \perp)$.*

A **polynomial embedding** *is an intent-preserving quasi-embedding into a polynomial context*

$$(K^n, K[x_1, \ldots, x_n], \perp)$$

over a given field K.

Note that the mappings which constituate the quasi-embedding do not have to be injective by definition. However, since they are compatible with the relations, they will be injective on purified contexts.

The following lemma is taken from [V] and shows that one inclusion of equation (1) holds for arbitrary embeddings.

Lemma 1. *A quasi-embedding (α, β) from (H, N, J) into (G, M, I) is an intent-preserving quasi-embedding if and only if $\beta(D^{JJ}) \subseteq (\beta(D))^{II} \cap \beta(N)$ holds for all $D \subseteq N$.*

Proof. We show that $\beta(D^{JJ}) \supseteq (\beta(D))^{II} \cap \beta(N)$ holds for all $D \subseteq N$ and all quasi-embeddings (α, β) from (H, N, J) into (G, M, I).

For all $D \subseteq N$ the equations $\alpha(D^J) = (\beta(D))^I \cap \alpha(H)$ and $\beta(D^{JJ}) = ((\beta(D))^I \cap \alpha(H))^I \cap \beta(N)$ hold. Since $(\beta(N))^I \cap \alpha(H) \subseteq (\beta(N))^I$, we have $((\beta(D))^I \cap \alpha(H))^I \supseteq (\beta(D))^{II}$ and therefore the desired inclusion.

Lemma 2. *Let (α, β) be an intent-preserving quasi-embedding from (H, N, J) into (G, M, I). Then the map ϕ_β defined by*

$$\phi_\beta(C, D) := ((\beta(D))^I, (\beta(D))^{II})$$

is a $\bigwedge$-preserving order embedding of $\underline{\mathfrak{B}}(H, N, J)$ into $\underline{\mathfrak{B}}(G, M, I)$.

Proof. [V], p.78.

The lemma tells us that the concept lattice of the context to be embedded can be visualized within the concept lattice of the context which contains the embedding. In particular, infima of concepts of (G, M, I) can be read of from $\mathfrak{B}(H, N, J)$.