The Basic Theorem on Labelled Line Diagrams of Finite Concept Lattices

Rudolf Wille

Technische Universität Darmstadt, Fachbereich Mathematik,
Schloßgartenstr. 7, D–64289 Darmstadt
wille@mathematik.tu-darmstadt.de

Abstract. This paper offers a mathematical analysis of labelled line diagrams of finite concept lattices to gain a better understanding of those diagrams. The main result is the *Basic Theorem on Labelled Line Diagrams of Finite Concept Lattices*. This Theorem can be applied to justify, for instance, the training tool "*CAPESSIMUS - A Game of Conceiving Concepts*" which has been created to support the understanding and the drawing of appropriate line diagrams of finite concept lattices.

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1 Introduction

For successfully applying *Formal Concept Analysis* in practice, well-readable diagrams of concept lattices are needed. Although the developed computer programs for drawing concept lattices are quite useful, they are still far from being satisfying, in particular, with respect to purpose support and adequate content representation. Human beings still have to take over the essential part of creating diagrammatic knowledge representations which are able to inspire, stimulate, and guide human thought. The purpose of this paper is to give some basic support for training humans in drawing *labelled line diagrams of finite concept lattices*.

It is important to realize that, in practice, labelled line diagrams of concept lattices have a *three-fold semantics*: a mathematical, a philosophical and a special purpose-oriented semantics. This triadic view is derived from the first level of *Peirce's classification of sciences* ([Pe92]; p.114) which lists the sciences in the order of abstractness: I. Mathematics II. Philosophy III. Special Sciences, where *Mathematics* is viewed as the most abstract science studying hypotheses exclusively and dealing only with potential realities, *Philosophy* is considered as the most abstract science dealing with actual phenomena and realities, while all

S.O. Kuznetsov and S. Schmidt (Eds.): ICFCA 2007, LNAI 4390, pp. 303–312, 2007.

other sciences are more concrete in dealing with special types of actual realities (see [GW06]; p.216).

Since, in practice, *labelled line diagrams of concept lattices* link mathematical, philosophical, and special thought, a well developed understanding of those diagrams is strongly desirable; in particular, users should be able to recognize whether a labelled line diagram represents a given concept lattice or not. This paper offers a mathematical analysis of labelled line diagrams to support the desired ability of understanding those diagrams. For that, *Section 2* recalls some basics of finite concept lattices and gives a proper proof for the Basic Theorem of Finite Concept Lattices, *Section 3* introduces a mathematization of line diagrams of finite bounded ordered sets, and *Section 4* formulates and proves the *Basic Theorem on Labelled Line Diagrams of Finite Concept Lattices.* In the final section, the Basic Theorem on Finite Labelled Line Diagrams is applied to support the conceptual training tool "*CAPESSIMUS - A Game of Conceiving Concepts*".

2 Basics of Finite Concept Lattices

Let us assume that the reader is familiar with the basic notions of Formal Concept Analysis as they are defined in [GW99]. From lattice theory the reader should particularly know that, in a finite lattice $\underline{L}$, each element of $\underline{L}$ is the supremum of $\vee$-*irreducible elements* and the infimum of $\wedge$-*irreducible elements* (see [DP02], p.55). In this paper, $J(\underline{L})$ denotes the set of all $\vee$-irreducible elements of $\underline{L}$ and $M(\underline{L})$ denotes the set of all $\wedge$-irreducible elements of $\underline{L}$. In a concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ of a finite context $\mathbb{K} := (G, M, I)$, each $\vee$-irreducible concept is of the form $\gamma g := (\{g\}'', \{g\}')$ for some $g \in G$ and each $\wedge$-irreducible concept is of the form $\mu m := (\{m\}', \{m\}'')$ for some $m \in M$, i.e., γG contains $J(\underline{\mathfrak{B}}(\mathbb{K}))$ and μM contains $M(\underline{\mathfrak{B}}(\mathbb{K}))$.

Now, we are prepared to formulate and to prove the finite case of the Basic Theorem on Concept Lattices (Part II) (cf. [GW99]; p.20):

Basic Theorem on Finite Concept Lattices (Part II). *A finite lattice $\underline{L}$ is isomorphic to the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ of a finite context $\mathbb{K} := (G, M, I)$ if and only if there exist mappings $\tilde{\gamma} : G \to \underline{L}$ and $\tilde{\mu} : M \to \underline{L}$ such that*

1. *$\tilde{\gamma}G$ contains $J(\underline{L})$,*
2. *$\tilde{\mu}M$ contains $M(\underline{L})$,*
3. *$gIm \iff \tilde{\gamma}g \leq \tilde{\mu}m$ for $g \in G$ and $m \in M$.*

Proof: Let ξ be an isomorphism from $\underline{\mathfrak{B}}(\mathbb{K})$ onto $\underline{L}$. If, for some $(A, B) \in \mathfrak{B}(\mathbb{K})$, $\bigvee\{\gamma g \mid g \in A\}$ is a $\vee$-irreducible element of $\underline{\mathfrak{B}}(\mathbb{K})$, there must exist an object $h \in A$ with $\gamma h = \bigvee\{\gamma g \mid g \in A\}$. Since in a finite lattice every element is the supremum of $\vee$-irreducible elements, it follows that $J(\underline{\mathfrak{B}}(\mathbb{K})) \subseteq \gamma G$ and hence $J(\underline{L}) = \xi J(\underline{\mathfrak{B}}(\mathbb{K})) \subseteq \xi\gamma G$. Dually, we obtain $M(\underline{L}) \subseteq \xi\mu M$. For $g \in G$ and $m \in M$ we have the following equivalences: $gIm \iff \gamma g \leq \mu m \iff \xi\gamma g \leq \xi\mu m$. Thus, defining $\tilde{\gamma} := \xi\gamma$ and $\tilde{\mu} := \xi\mu$ yields the conditions 1, 2, and 3.