Bipartite Ferrers-Graphs and Planar Concept Lattices

Christian Zschalig

Institut für Algebra, TU Dresden, Germany
Christian.Zschalig@tu-dresden.de

Abstract. There exists a close relation between the Ferrers-dimension of a context and the order dimension of the appropriate concept lattice [4]. Based on this fact we will introduce *Ferrers-Graphs* on contexts and show how they characterize planar concept lattices.

1 Introduction

In this work we will show how to decide whether a concept lattice is planar out of their contexts. We will introduce *Ferrers-graphs* [6] for that purpose. They are indicating which elements of the cross table are not in the same *Ferrers-relation* [4]. In case of this graph being bipartite we introduce a *left-relation on the context* based on its vertex classes. This relation will be used to define a *left-relation on the concept lattice* [7]. Since this relation is a strict order, the lattice is planar.

2 Preliminaries

All contexts considered in this work will be finite and reduced.

2.1 Ferrers-Graphs

In this part we want to remind some basics about Ferrers-relations and the Ferrers-dimension. We will introduce the notion of Ferrers-graphs and state the conjecture that will be proven in the course of that work.

Definition 1. *[4] A* Ferrers-relation F *is a relation* $F \subseteq A \times B$ *with*

$$a_1 F b_1 \wedge a_2 F b_2 \quad\Longrightarrow\quad a_1 F b_2 \vee a_2 F b_1.$$

The Ferrers-dimension $fdim(\mathbb{K})$ *of a context* $\mathbb{K} = (G, M, I)$ *is the smallest number of Ferrers-Relations* $F_t \subseteq G \times M,\ t \in T$*, whose intersection is equal to* I*, i.e.* $I = \bigcap_{t \in T} F_t$*.*

In a cross table representing a context $\mathbb{K} = (G, M, I)$ we notice that I is a Ferrers-relation if and only if the configuration depicted on the right does not occur.

	m_1	m_2
g_1	×	
g_2		×

S.O. Kuznetsov and S. Schmidt (Eds.): ICFCA 2007, LNAI 4390, pp. 313–327, 2007.

The inverse $\overline{F}$ of a Ferrers-relation is again a Ferrers-relation. Hence this Ferrers-dimension of a context $\mathbb{K} = (G, M, I)$ is the smallest number of Ferrers-relations covering the empty cells of its cross table [4], i.e. $\overline{I} := (G \times M) \setminus I = \bigcup_{t \in T} F_t$. The following theorem gives a connection to the *order dimension* of a lattice.

Theorem 1. *[4] Let $\mathbb{K}$ be a context. Then $fdim(\mathbb{K}) = dim(\underline{\mathfrak{B}}(\mathbb{K}))$.*

We know that a lattice is planar if and only if its order dimension is at most 2 (see Theorem 2). Hence the result already gives a nice characterization of contexts possessing planar concept lattices. Unfortunately the calculation of the Ferrers-dimension in general is $\mathcal{NP}$-complete [4].

Now we will introduce our notion of a *Ferrers-graph*. Its nodes are the empty cells of a context and its edges indicate which vertices can not belong to the same Ferrers-relation $\overline{F}$.

Definition 2. *[6] Let $R \subseteq A \times B$ be a relation. We define the* Ferrers-graph *$\Gamma(R)$ as follows:*

$$V(\Gamma(R)) := \overline{R} \qquad E(\Gamma(R)) := \{\{(a_1, b_2), (a_2, b_1)\} \mid (a_1, b_1), (a_2, b_2) \in R\}.$$

Let $\chi(\Gamma(I))$ is the chromatic number of $\Gamma(I)$. There is a conjecture claiming that

$$fdim(\mathbb{K}) = r \iff \chi(\Gamma(I)) = r$$

We will show in this work that this assertion holds for $r = 2$. It is easy to see that the first statement implies the second:

Lemma 1. *[5] For a context $\mathbb{K} = (G, M, I)$ the following implication holds:*

$$fdim(\mathbb{K}) = 2 \implies \Gamma(I) \text{ is bipartite.}$$

Proof. Since $fdim(\mathbb{K}) = 2$ there exist two Ferrers-relations F_1 and F_2 with $F_1 \cup F_2 = \overline{I} = V(\Gamma(I))$. Let (g_1, m_1) and (g_2, m_2) be elements of F_1. By Definition 1 we notice $g_1 \not I\, m_2$ or $g_2 \not I\, m_1$, i.e. $\{(g_1, m_1), (g_2, m_2)\} \notin E(\Gamma(I))$. Analogously we conclude that there exist no edges between elements of F_2. Hence $\Gamma(I)$ is bipartite with the vertex classes F_1 and $F_2 \setminus F_1$. $\square$

2.2 Conjugate Orders

Conjugate orders are a powerful tool to characterize planar lattices and ordered sets. Here we need that notion for introducing left-relations which are more convenient for our purpose.

Definition 3. *[3] Let $\underline{P} = (P, \leq)$ be an ordered set. The incomparability relation in $\underline{P}$ is denoted by $\|$.*

1. *We call L_c* conjugate relation *if $L_c \cup {L_c}^{-1} = \|$.*
2. *We call L_c* conjugate order *if additionally L_c is a strict order.*