The Hypervolume Indicator Revisited: On the Design of Pareto-compliant Indicators Via Weighted Integration

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Abstract. The design of quality measures for approximations of the Pareto-optimal set is of high importance not only for the performance assessment, but also for the construction of multiobjective optimizers. Various measures have been proposed in the literature with the intention to capture different preferences of the decision maker. A quality measure that possesses a highly desirable feature is the hypervolume measure: whenever one approximation completely dominates another approximation, the hypervolume of the former will be greater than the hypervolume of the latter. Unfortunately, this measure—as any measure inducing a total order on the search space—is biased, in particular towards convex, inner portions of the objective space. Thus, an open question in this context is whether it can be modified such that other preferences such as a bias towards extreme solutions can be obtained. This paper proposes a methodology for quality measure design based on the hypervolume measure and demonstrates its usefulness for three types of preferences.

1 Motivation

Using the hypervolume of the dominated portion of the objective space as a measure for the quality of Pareto set approximations has received more and more attention in recent years. The reason is that this measure has two important advantages over other set measures:

1. It is sensitive to any type of improvements, i.e., whenever an approximation set A dominates another approximation set B, then the measure yields a strictly better quality value for the former than for the latter set [23].
2. As a result from the first property, the hypervolume measure guarantees that any approximation set A that achieves the maximally possible quality value for a particular problem contains all Pareto-optimal objective vectors [5].

So far, this is the only measure known in the literature on evolutionary multicriterion optimization that possesses these properties.

The hypervolume measure—or *hypervolume indicator* [23]—was first proposed and employed in [21,22] where it was denoted as 'size of the space covered'; later,

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also other terms such as 'hyperarea metric' [14], 'S-metric' [18], and 'Lebesgue measure' [11,5] were used. On the one hand, the hypervolume indicator is meanwhile among the most popular measures for the performance assessment of multiobjective optimizers and in this context it has been subject to several theoretical investigations [8,5,23,15]. On the other hand, there are some studies that discuss the usage of this measure for multiobjective search [10,20,4] and in particular the issue of fast hypervolume calculation has been a focus of research recently [16,17,6,1].

Despite the aforementioned advantages of the hypervolume indicator, it inevitably has its biases. There is some freedom with respect to the choice of the reference point, but nevertheless it represents only one particular class of preference information that may not be appropriate in specific situations. This discussion directly leads to the question of whether it is possible to design quality measures that (i) share the two above properties of the hypervolume indicator, while (ii) standing for a different type of preferences of the decision maker. The fact that besides the hypervolume no other measures with these properties are known indicates that the formalization of *arbitrary* preferences in terms of a quality measure may be difficult. However, not being aware of such measures does not imply that such indicators do not exist.

This paper presents a first step to tackle this issue: it demonstrates that novel quality measures with the aforementioned properties can be designed and proposes a general design methodology on the basis of the hypervolume indicator. In detail, the key contributions are:

- A generalized definition of the hypervolume indicator using attainment functions [2] that can be used for any type of dominance relation;
- A weighted-integration approach to directly manipulate and control the influence of certain regions in the objective space for the hypervolume indicator;
- Three new example set measures for biobjective problems that provide the same sensitivity as the hypervolume indicator, but represent different types of preference information: (i) the preference of extreme solutions, (ii) the preference of predefined reference points, and (iii) bias towards one of the objectives.

The usefulness of the methodology and the three proposed measures is demonstrated on selected test problems.

2 Mathematical Framework

2.1 Preliminaries

Without loss of generalization, we consider a maximization problem with n objective functions $f_i : X \to (0,1)^n$, $1 \leq i \leq n$. Requiring the objective values to lay between 0 and 1 instead of using $\mathbb{R}^n$ as objective space simplifies the following discussions, but does not represent a serious limitation as there exists a bijective mapping from $\mathbb{R}$ into the open interval $(0,1) \subset \mathbb{R}$. The objective