Chapter X
Lattices and packings in higher dimensions

X.1. Lattices and packings associated with dimension 3

A lattice in $\mathbb{R}^3$ is a $\Lambda$ that can be written as the set of integer combinations of three linearly independent vectors $\{a, b, c\}$, say $\Lambda = \mathbb{Z} \cdot a + \mathbb{Z} \cdot b + \mathbb{Z} \cdot c$. As in Sect. IX.4, two Euclidean invariants are immediately associated with a lattice; they are practically dictated when we seek to pack balls of like radius in the densest possible way, thus the most economical for practical life; see more in Sect. X.4. If we want to place the centers of these balls of given radius $R$ at the vertices of a $\Lambda$, then it is necessary that $||\lambda - \mu|| > 2R$ for all the $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$, whence the notion of minimal norm of a lattice:

$$\text{NorMin}(\Lambda) = \inf\{||\lambda - \mu|| : \lambda, \mu \in \Lambda \text{ and } \lambda \neq \mu\}.$$  

The volume of a lattice is the absolute value $|\det(a, b, c)|$ of the determinant of any basis $\{a, b, c\}$ of $\Lambda$. Then the density of the lattice will be, as in (IX.4.1):

$$\text{Dens}(\Lambda) = \frac{\pi (\text{NorMin}(\Lambda))^3}{6 \text{Vol}(\Lambda)},$$

in other words, the quotient of the volume of the ball of radius $R$ (i.e. $4/3\pi R^3$), for $R = \frac{1}{2} \text{NorMin}(\Lambda)$ (the maximum value of $R$), to the volume of the basic parallelepiped of the lattice.

Of course, our quantity being invariant under change of scale, we suppose henceforth that $R = \frac{1}{2}$, i.e. that $\text{NorMin} = 1$. What lattice yields the largest value for the density? The result was conjectured by Seeber in 1831 and proved by Gauss the same year. Gauss wanted to answer this question, because it had come up in number theory in studying quadratic forms with integer values, well before the Minkowski’s general theories; see Sect. IX.3. This is a problem of elementary geometry concerning the tetrahedron in $\mathbb{R}^3$ formed by a basis $(0, a, b, c)$: which is the one of smallest volume under the condition that all 6 edges have length greater or equal to unity. Readers will have guessed the answer: it is the regular tetrahedron and we leave the corresponding proof (which isn’t so easy) to them; see also a little further below or Sect. VIII.7.

What does our optimal lattice look like? A small difficulty is that there are two distinct ways of seeing it, of realizing it. Even though the equivalence is very easy
(and a good exercise in spatial visualization), its realization is important for the following section and can be seen in the figures below:

Fig. X.1.1. A\textsubscript{3} and D\textsubscript{3}

We can view the two associated ball packing, seemingly quite different, in Fig. X.1.2.

Fig. X.1.2.

We don’t give coordinates of a basis for the lattice A\textsubscript{3}, leaving this for readers to do, but in the fastest way, by placing themselves in dimension 4 and observing that the set \( \Lambda \) is the intersection of \( \mathbb{Z}^4 \subset \mathbb{R}^4 \) with the hyperplane with equation \( x_1 + x_2 + x_3 + x_4 = 0 \). The second figure suggests the basis \( \{(1, 0, 0), (0, 1, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})\} \), but this representation isn’t pleasant because it isn’t very symmetric; after a change of scale by the factor \( \sqrt{2} \) we replace it by \( \{(1, 1, 0), (1, -1, 0), (1, 0, 1)\} \). But above all we remark that the lattice generated can be defined as \( \{(x, y, z) \in \mathbb{Z}^3 : x+y+z \text{ even}\} \).

We emphasize yet again that these two definitions lead to identical lattices as regards their Euclidean structure (see Fig. X.1.3) and so we obtain one and only one optimal packing of balls of the same radius, of the same lattice type.

Owing to crystallographers, this lattice — and of course any similar lattice — is called the cubic face centered lattice. It’s the lattice that appears in the structure of carbon, but some atoms need to be added to the basic cubes. For pure mathematicians, it’s denoted A\textsubscript{3} or D\textsubscript{3}. We will see why in Sect. X.4. Thus: