Chapter III

The sphere by itself: can we distribute
points on it evenly?

III.1. The metric of the sphere and spherical trigonometry

As we shall see throughout this chapter, the geometry of the “ordinary sphere” $S^2$ — two dimensional in a space of three dimensions — harbors many pitfalls. It’s much more subtle than we might think, given the nice roundness and all the symmetries of the object. Its geometry is indeed not made easier — at least for certain questions — by its being round, compact, and bounded, in contrast to the Euclidean plane. Sect. III.3 will be the most representative in this regard; but, much simpler and more fundamental, we encounter the “impossible” problem of maps of Earth, which we will scarcely mention, except in Sect. III.3; see also 18.1 of [B]. One of the reasons for the difficulties the sphere poses is that its group of isometries is not at all commutative, whereas the Euclidean plane admits a commutative group of translations. But this group of rotations about the origin of $\mathbb{E}^3$ — the orthogonal group $O(3)$ — is crucial for our lives. Here is the place to quote (Gromov, 1988b):

“O(3) pervades all the essential properties of the physical world. But we remain intellectually blind to this symmetry, even if we encounter it frequently and use it in everyday life, for instance when we experience or engender mechanical movements, such as walking. This is due in part to the non commutativity of $O(3)$, which is difficult to grasp.”

Since spherical geometry is not much treated in the various curricula, we permit ourselves at the outset some very elementary recollections, which are treated in detail in Chap. 18 of [B]. Typically we will deal with the sphere $S^2$ of radius 1 centered at the origin of the Euclidean space $\mathbb{E}^3$, but all spherical geometries are the same within a change of scale. The interesting distance between two points $p$ and $q$ of $S^2$ isn’t their distance in $\mathbb{E}^3$ but that on $S^2$, i.e. the length of a shortest path on $S^2$ that joins $p$ to $q$. We are dealing here with the metric that is called intrinsic or internal, as opposed to the induced metric that is the distance between these points in $\mathbb{E}^3$ and evidently without much interest for Earth’s inhabitants: digging tunnels is expensive. It’s classical that this path is unique and that is the arc of a great circle that joins $p$ to $q$, which can be proved rigorously by a symmetry argument, or here with the fundamental formula (III.1.1) below. Note that in saying the arc of a great circle we necessarily exclude antipodes, for which all the great circles starting from the one arrive at the other after a distance $\pi$. We let $d(p,q)$ denote this distance, which varies from 0 to $\pi$. It is given in terms of the scalar product by the formula:
\[ \cos(d(p, q)) = p \cdot q. \] But it may also be regarded as the angle through which we see the points \( p \) and \( q \) from the origin \( O \). We can say, in an entirely equivalent manner, that we are dealing here with seeing the sphere as a set of half-lines emanating from the origin, which is crucial in descriptive astronomy: the sphere is the celestial vault and there is an essential need of the formulas that will follow. As astronomy is very old, these formulas date from several centuries ago, but we will read them on \( S^2 \), not as would astronomers.

![Fig. III.1.1.](image)

We remain on \( S^2 \). It is clear that when we have three points \( p, q, r \) of \( S^2 \) we may speak — in analogy to Euclidean geometry — of the spherical triangle \( \{ p, q, r \} \).

Now if we know the distances of \( d(p, q) \) and \( d(p, r) \), as well as the angle between the sides that emanate from \( p \), we sense in advance that the distance of \( d(q, r) \) is determined, thus theoretically calculable. The formula that provides this distance is called the first fundamental formula of spherical trigonometry. Let us call \( a, b, c \) the lengths (distances) of the sides of \( \{ p, q, r \} \), and \( \alpha, \beta, \gamma \) its angles (between 0 and \( \pi \)). Then we always have:

\[
(III.1.1) \quad \cos a = \cos b \cos c + \sin b \sin c \cos \alpha.
\]

The proof is a direct application of the definitions and of the scalar product, whose neglect in school mathematics explains why this formula isn’t better known to students. It was prohibited in the French lycées until about 1950, which explains the incredible contortions in treating the inequality of days and nights, which was part of the required program. From it we can deduce — cautioning that the value of the sine does not determine the angle unambiguously — all the formulary given in detail in [B]. The practical importance is considerable; this or analogous formulas provide the solution to the following problem relating to three directions (or three half-lines) in space. Such a configuration is associated, as is the corresponding spherical triangle, with six numbers: the three angles between each pair of lines, and also the angles between the planes determined by pairs of these lines. This play with formulas allows us to calculate all these elements as a function of only three of them, but with some precaution. Think about the “dubious” case of equality of triangles. Readers