Elements of Stochastic Calculus via Regularization

A la mémoire de Paul André Meyer

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Summary. This paper first summarizes the foundations of stochastic calculus via regularization and constructs through this procedure Itô and Stratonovich integrals. In the second part, a survey and new results are presented in relation with finite quadratic variation processes, Dirichlet and weak Dirichlet processes.

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1 Introduction

Stochastic integration via regularization is a technique of integration developed in a series of papers by the authors starting from [46], continued in [47, 48, 49, 50, 45] and later carried out by other authors, among them [51, 12, 13, 55, 54, 56, 58, 17, 16, 18, 19, 24]. Among some recent applications to finance, we refer for instance to [32, 4].

This approach constitutes a counterpart of a discretization approach initiated by Föllmer [20] and continued by many authors, see for instance [2, 22, 15, 14, 11, 23].

The two theories run parallel and, at the axiomatic level, almost all the results we obtained via regularization can essentially be translated in the language of discretization.
The advantage of using regularization lies in the fact that this approach is natural and relatively simple, and easily connects to other approaches. We now list some typical features of stochastic calculus via regularization.

- Two fundamental notions are the quadratic variation of a process, see Definition 2 and the forward integral, see Definition 1. Calculus via regularization is first of all a calculus related to finite quadratic variation processes, see Section 4. Itô integrals with respect to continuous semimartingales can be defined through forward integrals, see Section 3; this makes classical stochastic calculus appear as a particular instance of calculus via regularization. Let the integrator be a classical Brownian motion $W$ and the integrand a measurable adapted process $H$ such that $\int_0^T H_t^2 dt < \infty$ a.s., where a.s. means almost surely. We will show in Section 3.5 that the forward integral $\int_0^T HdW$ coincides with the Itô integral $\int_0^T HdW$. On the other hand, the discretization approach constitutes a sort of Riemann–Stieltjes type integral and only allows integration of processes that are not too irregular, see Remark 14.

- Calculus via regularization constitutes a bridge between noncausal and causal calculus operating through substitution formula, see Section 3.6. A precise link between forward integration and the theory of enlargement of filtrations may be given, see [47]. Our integrals can be connected to the well-known Skorohod type integrals, see again [47].

- With the help of symmetric integrals a calculus with respect to processes with a variation higher than 2 may be developed. For instance fractional Brownian motion is the prototype of such processes.

- This stochastic calculus constitutes somehow a barrier separating the pure pathwise calculus in the sense of T. Lyons and coauthors, see e.g., [36, 35, 31, 28], and any stochastic calculus taking into account an underlying probability, see Section 6.

This paper will essentially focus on the first item.

The paper is organized as follows. First, in Section 2, we recall the basic definitions and properties of forward, backward, symmetric integrals and covariations. Justifying the related definitions and properties needs no particular effort. A significant example is the Young integral, see [57]. In Section 3 we redefine Itô integrals in the spirit of integrals via regularization and we prove some typical properties. We essentially define Itô integrals as forward integrals in a subclass and we then extend this definition through functional analysis methods. Section 4 is devoted to finite quadratic variation processes. In particular we establish $C^1$-stability properties and an Itô formula of $C^2$-type. Section 5 provides some survey material with new results related to the class of weak Dirichlet processes introduced by [12] with later developments discussed by [24, 7]. Considerations about Itô formula under $C^1$-conditions are discussed as well.