Some Issues in Relativistic Spacetime Theories

5.1 Reference Frames on Relativistic Spacetimes

In this chapter \[ M = (M, g, D, \tau, \uparrow) \] denotes a Lorentzian spacetime and \[ M = (M, \eta, D, \tau, \uparrow) \] denotes Minkowski spacetime. We adopt, as before, natural units such that the value of the velocity of light is \( c = 1 \).

**Proposition 313** Let \( Q \in \text{sec} TU \subset \text{sec} TM \) be a timelike vector field such that \( g(Q, Q) = 1 \). Then, there exist, in a coordinate neighborhood \( U \), three spacelike vector fields \( e_i \) which together with \( Q \) form an orthogonal moving frame for \( x \in U \).

**Proof.** Suppose that the metric of the manifold in a chart \( (U, \phi) \) with coordinate functions \( \{x^\mu\} \) is \( g = g_{\mu\nu} dx^\mu \otimes dx^\nu \). Let \( Q = (Q^\mu \partial / \partial x^\mu) \in \text{sec} TM \) be an arbitrary reference frame. Then, \( g_{\mu\nu}(x)Q^\mu Q^\nu = 1 \). Put \( Q_\mu = g_{\mu\nu}Q^\nu \) and define

\[
\theta^0 = \alpha_Q = g(Q, \cdot) = Q_\mu dx^\mu,
\gamma_{\mu\nu} = g_{\mu\nu} - Q_\mu Q_\nu.
\] (5.1)

Then the metric \( g \) can be written due to the hyperbolicity of the manifold as

\[
g = \theta^0 \otimes \theta^0 - \sum_{i=1}^{3} \theta^i \otimes \theta^i,
- \sum_{i=1}^{3} \theta^i \otimes \theta^i = \gamma_{\mu\nu}(x) dx^\mu \otimes dx^\nu.
\] (5.2)

Now, call \( e_0 = Q \) and take \( e_a \) such that \( \theta^a(e_b) = \delta^a_b \). It follows immediately that \( g(e_a, e_b) = \eta_{ab} \), \( a, b = 0, 1, 2, 3 \). \( \square \)

Before we proceed we need to know precisely how the metric \( g \) relates tangent space magnitudes to magnitudes on the manifold. Let \( \sigma : \mathbb{R} \supset I \rightarrow M \), be a smooth curve, i.e. \( \sigma \) is \( C^0 \) and piecewise \( C^1 \). We denote the inclusion
function \( I \to \mathbb{R} \) by \( u \), and the distinguished vector field on \( I \) by \( \frac{d}{du} \). For each \( u \in I \), \( \sigma_{*u} \) denotes the tangent vectors at \( \sigma(u) \in M \). Thus,

\[
\sigma_{*u} = \left[ \sigma_{*} \left( \frac{d}{du} \right) \right]_{\sigma(u)} \in T_{\sigma(u)}M ,
\]

where \( \sigma_{*} \) denotes the derivative mapping of the mapping \( \sigma \).

**Definition 314** A curve is said timelike (respectively lightlike, or respectively spacelike) when for all \( u \in I \), \( g(\sigma_{*u}, \sigma_{*u}) > 0 \) (respectively \( g(\sigma_{*u}, \sigma_{*u}) = 0 \), or respectively \( g(\sigma_{*u}, \sigma_{*u}) < 0 \)).

**Definition 315** The path length between events \( e_1 = \sigma(a) \) and \( e_2 = \sigma(b) \) along the curve \( \sigma \), such that for all \( u \in I \), \( g(\sigma_{*u}, \sigma_{*u}) \) has the same signal at all points along \( \sigma(u) \) is the quantity

\[
\int_{a}^{b} \left[ |g(\sigma_{*u}, \sigma_{*u})| \right]^{\frac{1}{2}} .
\]

Observe that taking the point \( \sigma(a) \) as a reference point we can use (5.4) to define a function with domain \( \sigma(I) \) by

\[
s : \sigma(I) \to \mathbb{R}, \quad s(\sigma(u)) := s(u) = \int_{r}^{u} \left[ |g(\sigma_{*u'}, \sigma_{*u'})| \right]^{\frac{1}{2}} .
\]

With (5.5) we can calculate the derivative of the function \( s \) (after introducing a local chart \( \{x^\mu\} \) covering the domain of interest)

\[
\frac{ds}{du} = \left[ |g(\sigma_{*u'}, \sigma_{*u'})| \right]^{\frac{1}{2}} = \left[ \left| g_{\mu\nu} \frac{d\xi^\mu}{du} \right| \right]^{\frac{1}{2}} .
\]

From (5.6) old books on differential geometry and almost all books on General Relativity write the equation

\[
(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu ,
\]

which is supposed to represent the square of the length of the ‘infinitesimal’ arc determined by the coordinate displacement

\[
x^\mu \circ \sigma(r) \to x^\mu \circ \sigma(r) + \frac{d\xi^\mu}{du}(r)\varepsilon ,
\]

where \( \varepsilon << 1 \) is an ‘infinitesimal’ number.

The above mathematically correct notation is somewhat cumbersome, and when no confusion arises \( x^\mu \circ \sigma(r) \) is denoted simply by \( x^\mu(r) \).

We recall (Remark 547) that a moving frame\textsuperscript{1} for \( U \subset M \) is a basis of vector fields for \( \mathcal{H}(U) \) (the module of vector fields at \( U \)). An orthonormal frame for \( U \subset M \) is a basis of orthonormal vector fields for \( \mathcal{H}(U) \).

\textsuperscript{1} M is assumed as part of the spacetime structure \( \mathfrak{M} = (M, g, D, \tau_g, \uparrow) \).