General Classification of (1,2) Parametric Surfaces in $\mathbb{P}^3$

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Summary. Patches of parametric real surfaces of low degrees are commonly used in Computer Aided Geometric Design and Geometric Modeling. However the precise description of the geometry of the whole real surface is generally difficult to master, and few complete classifications exist.

Here we study surfaces of bidegree (1,2). We present a classification and a geometric study of parametric surfaces of bidegree (1,2) over the complex field and over the real field by considering a dual scroll. We detect and describe (if it is not void) the trace of self-intersection and singular locus in the system of coordinates attached to the control polygon of a patch (1,2) in the box $[0; 1] \times [0; 1]$.

6.1 Introduction

We consider a polynomial mapping of bidegree (1,2):

$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$$

given by a matrix $A = (a_{ij}), i = 1, \ldots, 4, j = 1, \ldots, 6$ of maximal rank 4 such that:

$$\Phi = t^4(\Phi_1, \Phi_2, \Phi_3, \Phi_4) \quad \text{and} \quad \Phi = A^4(tu^2, tuv, tv^2, su^2, svu, sv^2) \quad (1)$$

where $(t : s), (u : v)$ are a system of coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. The base field is $K = \mathbb{C}$ or $\mathbb{R}$. Then $S = \text{Im}(\Phi) \subset \mathbb{P}^3$ is a parametric surface of bidegree (1,2) and $\Phi$ is a parameterization of $S$.

Similarly, one defines surfaces of bidegree $(m, n)$; patches of these surfaces are often used in C.A.G.D and Solid Modeling especially for the bi-cubics $m = n = 3$.

Our aim is to classify the applications $\Phi$ of bidegree (1,2) while the base field is $\mathbb{R}$ or $\mathbb{C}$ up to a change of projective coordinates in the source space $\mathbb{P}^1 \times \mathbb{P}^1$ and in the target space $\mathbb{P}^3$. In a previous article [13] we described the generic complex case and the geometry of the corresponding surfaces. Then, the parameterization of $\Phi$ is equivalent to a parameterization, we called "normal form" and denoted by $NF(a, b)$:
where \( a \) and \( b \) are two complex parameters different from 0 and 1. Moreover, if \((a, b) \neq (a', b')\) then \( NF(a, b) \) is not equivalent to \( NF(a', b') \). We say that \((a, b)\) is a couple of moduli for this classification.

In this article we study the real generic cases and the non generic cases. The surfaces \( S \) defined by (1) are ruled surfaces which admit an implicit equation in \( \mathbb{P}^3 \) of degree at most 4. These surfaces were studied extensively in the 19th century by great mathematicians: Cremona [7], Cayley [3], Segre [28]; one finds a synthesis of theirs results and extensions in the books of Salmon [25] and of Edge [12]. From 1930, the mainstream of algebraic geometry concentrated on the study of varieties up to birational equivalence and with more conceptual (and less effective) tools. However, applications in C.A.G.D and Solid Modeling showed the necessity of revisiting the geometry of parametrized curves and surfaces of small degrees and bidegrees. An article of Coffman and al. [6] is a model of this kind of work: it revisited and completed the classification of parameterized surfaces of total degree 2 (started by Steiner in 1850). The ruled surfaces of implicit degree 4 are more complicated and have more diversity. In the 19th century the focus was not on the classification of parameterizations but rather on the geometric property and the calculation of certain invariants as well as on the obtaining of lists of implicit equations which are dependent of many parameters. A presentation of these classification results over the complex field related to rational (1,2)-Bézier surfaces with the description of the behaviour in presence of base-points, but without any description of the singularities, was provided by W.L.F. Degen [9]. A more complete classification over the real field, describing also the possible singularities was provided by S. Zube in [30] and [31]. Here we briefly review all these results, then we provide a new presentation based on the study of the dual scroll and the consideration of the tangent planes to all conics of the surface. We provide normal forms of the parameterizations and relate them to geometric data of the surface. We also consider the problem of defining classifying spaces which express the proximity with respect to deformations of these objects. Our article is organized as following:

In section 2, we recall some results of the 19th century, we follow the syntax given by Edge [12] in 1931, then we distinguish different types of parametric surfaces and we concentrate on the surfaces of bidegree (1,2). In section 3, we present our method of classification and introduce a scroll surface in the dual space which will be used to find the moduli. This variety is different from those used by the geométricians of 19th century but similar to the ones considered in [23]. In section 4, we recall the results obtained in [13] for generic complex case and extend them to the real setting. In section 5, we classify the intersections of a scroll (1,2) of \( \mathbb{P}^5 \) and a 3-projective plane or equivalently to the intersections of two curves of bidegree (1,2) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Then we apply these results to the classification of parametric surfaces (1,2). In section 6, we provide simple formulae to describe the critical points in the system of coordinates attached to the control polygon of a patch (1,2). We detect and describe the trace of the pre-images of the self-intersection and singular locus in the box \([0, 1] \times [0, 1] \).