Effective-field theory and the nuclear many-body problem

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Abstract. We review many-body calculations of the equation of state of dilute neutron matter in the context of effective-field theories of the nucleon-nucleon interaction.

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1 Introduction

One of the central problems of nuclear physics is to calculate the properties of nuclear matter starting from the two-body scattering data and the binding energies of few-body bound states [1,2]. The nuclear-matter problem is notoriously difficult. Some of the problems that are often mentioned are:

- the large short-range repulsive core in the nucleon-nucleon interaction,
- the large scattering length in the $^1S_0$ channel, the small binding energy of the deuteron, and the small saturation density,
- the need to include three (and possibly four) body forces,
- the need to include non-nucleonic degrees of freedoms, such as isobars, mesons, quarks, etc.

Ever since the discovery of QCD the classic nuclear-matter problem has evolved into the broader question of how the properties of nuclear matter are related to the parameters of the QCD, the QCD scale parameter and the masses of the light quarks.

Over the last couple of year much progress has been made in understanding these kinds of questions in the case of nuclear two- and three-body bound states [3]. Using effective-field theory methods it was shown that:

- the short-range behavior of the nuclear force is not observable. Using the renormalization group the short-distance behavior can be modified without changing low-energy scattering data and binding energies [4,5];
- effective field theories can accommodate the large scattering lengths in the nucleon-nucleon system [6,7].

The scattering lengths depend sensitively on the quark masses, see fig. 1, and the large value of $a(\,^1S_0\,)$ observed in nature appears to be accidental [8,9];

- a local three-body force is necessary to renormalize the two-body force already at leading order\(^1\) [10]. As a consequence, one cannot predict three-body binding energies based on two-body scattering data alone;
- non-nucleonic degrees of freedom, quark effects, relativistic effects etc. can be absorbed in local operators.

Effective-field theories have also achieved remarkable quantitative success in describing the available nucleon-nucleon scattering data below the pion production threshold [11,12]. The long-term goal is to achieve a similar

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\(^1\) An exception is the original Weinberg scheme in which three- and four-body forces are considered higher-order corrections.
qualitative and quantitative understanding of the nuclear many-body problem.

In this contribution we shall study a simple limiting case of the nuclear-matter problem. We shall concentrate on pure neutron matter at densities significantly below the nuclear-matter saturation density. The neutron-neutron scattering length is \( a_{\text{nn}} = -18 \text{ fm} \) and the effective range is \( r_{\text{nn}} = 2.8 \text{ fm} \). This means that there is a range of densities for which the inter-particle spacing is large compared to the effective range but small compared to the scattering length. Neutron matter in this regime exhibits interesting universal properties. We are interested in the limit \( (k_F a_{\text{nn}}) \to \infty \) and \( (k_F r_{\text{nn}}) \to 0 \), where \( k_F \) is the Fermi momentum. From a dimensional analysis it is clear that the energy per particle at zero temperature has to be proportional to the energy per particle of a free Fermi gas at the same density:

\[
\frac{E}{A} = \xi \left( \frac{E}{A} \right)_0 = \frac{3}{5} \left( \frac{k_F^2}{2m} \right).
\]

The constant \( \xi \) is universal, i.e. independent of the details of the system. Similar universal constants govern the magnitude of the gap in units of the Fermi energy and the equation of state at finite temperature.

Universality also implies that the properties of this system can be studied using atoms rather than nuclei. The scattering length of certain fermionic atoms can be tuned using Feshbach resonances, see [15] for a review. A small negative scattering length corresponds to a weak attractive interaction between the atoms. This case is known as the BCS limit. As the strength of the interaction increases the scattering length becomes larger. It diverges at the point where a bound state is formed. The point \( a = a_c \) is called the unitarity limit, since the scattering cross-section saturates the s-wave unitarity bound \( \sigma = 4\pi / k^2 \). On the other side of the resonance the scattering length is positive. In the BEC limit the interaction is strongly attractive and the fermions form deeply bound molecules.

\section{2 Numerical calculations}

The calculation of the dimensionless quantity \( \xi \) is a non-perturbative problem. In this section we shall describe an approach based on lattice field theory methods. The physics of the unitarity limit is captured by an effective Lagrangian of point-like fermions interacting via a short-range interaction. The Lagrangian is

\[
\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2m} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2.
\]

The standard strategy for dealing with the four-fermion interaction is to use a Hubbard-Stratonovich transformation. The partition function can be written as [16]

\[
Z = \int Ds Dc Dc^\dagger \exp \left[ -S \right],
\]

where \( s \) is the Hubbard-Stratonovich field and \( c \) is a Grassmann field. \( S \) is a discretized Euclidean action:

\[
S = \sum_{n,i} \left[ e^{-i\mathbf{k}_F a_{\text{nn}} (n)} c_i^\dagger (n) c_i (n+0) \right]
- e^{-1} \mathcal{C}^{\text{at}}(n) + C^{\text{at}}(n) \big( 1 - 6m \big) c_i^\dagger (n) c_i (n)
- h \sum_{n,\delta_2,\delta_1} \left[ c_i^\dagger (n) c_i (n + \delta_2) + c_i^\dagger (n) c_i (n - \delta_1) \right]
+ \frac{1}{2} \sum_n \sigma^2 (n).
\]

Here \( i \) labels spin and \( n \) labels lattice sites. Spatial and temporal unit vectors are denoted by \( \delta_2 \) and \( \delta_0 \), respectively. The temporal and spatial lattice spacings are \( b_t \) and \( b \). The dimensionless chemical potential is given by \( \mu = \mu_b \). We define \( \alpha_i \) as the ratio of the temporal and spatial lattice spacings and \( \hbar = \alpha_i / (2m) \). Note that for \( C_0 < 0 \) the action is real and standard Monte Carlo simulations are possible.

The four-fermion coupling is fixed by computing the sum of all two-particle bubbles on the lattice. Schematically,

\[
\frac{m}{4\pi a} = \frac{1}{C_0} + \sum_p \frac{1}{E_p},
\]

where the sum runs over discrete momenta on the lattice and \( E_p \) is the lattice dispersion relation. A detailed discussion of the lattice regularized scattering amplitude can be found in [17,18,16]. For a given scattering length \( a \) the four-fermion coupling is a function of the lattice spacing. The continuum limit correspond to taking the temporal and spatial lattice spacings \( b_t, b \) to zero

\[
b_t \mu \to 0, \quad b_t^{1/3} \to 0,
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Equation of state of pure neutron matter at \( T = 4 \text{ MeV} \) from lattice simulations of an effective-field theory. Figure taken from Lee and Schäfer [16]. The curves labeled fc, f1, f2 show results for a free gas on the lattice and in the continuum, the curves labeled b1, b2 show ladder sums, and s1, s2 are numerical results on different lattices. We also compare to the variational results of Friedman and Pandharipande (FP).}
\end{figure}