Stationary states, Fluctuation-Dissipation Theorem and effective temperature in a turbulent von Karman flow

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A yet unanswered question in statistical physics is whether stationary out-of-equilibrium systems share any resemblance with classical equilibrium systems. A good paradigm to explore this question is offered by turbulent flows. Incompressible flows subject to statistically stationary forcing generally reach a kind of equilibrium (in the statistical sense), independent of the initial conditions. Description of turbulence with tools borrowed from statistical mechanics is a long-standing dream, starting with Onsager. In 2D, equilibrium states of the Navier-Stokes equations have been classified through statistical mechanics principle by Robert and his collaborators [6; 1]. More recent advances have been done for 3D axisymmetric flows (an intermediate situation between 2D and 3D) by Leprovost et al. [2]. In the following, we present results obtained within this framework for a von Kármán flow.

1 Axisymmetric steady states

We consider the Euler limit of a turbulent flow, in which both forcing and dissipation are removed. In this limit, an axisymmetric flow is characterized by a number of global conserved quantities like energy and helicity. In addition, owing to the symmetry, there is conservation of angular momentum along a velocity line resulting in a Liouville theorem and additional global conserved quantities as Casimirs of angular momentum. This allows the definition of a mixing entropy, and derivation of Gibbs states of the problem through a procedure of maximization of the mixing entropy under constraints of conservation of the global quantities. From the Gibbs state, one can derives general identities characterizing the steady states [2; 4]: \( \sigma = F(\Psi) \), \( \xi - \frac{\int F r}{r} = G(\Psi) \), with \( \xi = \omega_\theta / r \), where \( F \) and \( G \) are arbitrary functions linked with conservation laws of the system, \( \sigma \) is the angular momentum, \( \psi \) the stream function.
and $\omega_\theta$ the azimuthal vorticity. Furthermore, $r^{-1}\partial_r \left(r^{-1}\partial_r \psi \right) + r^{-2}\partial^2_z \psi = -\xi$. We have verified the existence of similar relations for steady states in an experimental von Kármán flow for different impellers and a wide range of Reynolds number (from 100 to 314000). Details of the experimental setup are given in Ref. [5; 3]. A representative result is provided on figure (1). One sees that while data on the whole vessel display significant scatter, preventing the outcome of a well-defined $F$, the data far from boundaries and impellers gather onto a cubic-shaped function fitted by a two parameters cubic $F(\Psi) = p_1\Psi + p_3\Psi^3$. This fit is then used to obtain $G$. Furthermore, we found that they depend

![Graphs showing data for F and G](image)

**Fig. 1.** $F$ (left) and $G$ (right) at $Re = 5 \times 10^5$. The light gray dots corresponds to the whole flow, the dark gray crosses to 50% of the flow: $r/R \in [-0.6; 0.6]$, $z/R \in [-0.4; 0.4]$ and the solid line to a cubic fit.

on the impellers and on Reynolds number: with other impellers $F$ (resp. $G$) tend to be linear (resp. equal to zero) as Reynolds number is increased [4]. This can be interpreted as a signature of a Beltramiization of the flow (i.e. a depletion of nonlinearities).

## 2 Fluctuations

In the Beltrami limit, the Gibbs states of the Euler equation can be used to derive two relations between steady solutions and their fluctuations:

$$
\langle (\mu \sigma)^2 \rangle - \mu \sigma^2 = -\frac{\delta \mu \sigma}{\delta \xi} = \frac{\mu^2}{\beta} \frac{1}{r^2},
$$

$$
\bar{\xi}^2 - \bar{\epsilon}^2 = -\frac{\delta \bar{\xi}}{\delta \mu \bar{\sigma}} = \frac{\beta}{\mu^2} \frac{1}{r^2}.
$$

where $\beta^{-1}$ is an effective temperature and $\mu$ is a vortical susceptibility (the Lagrange parameter associated to the energy and helicity respectively). We have tested these two relations for fluctuations in our experiment. Results are provided on figure (2). Once more the relations are satisfied in the core of