Electromagnetic and Weak Radiative Corrections

4.1 $g - 2$ in Quantum Electrodynamics

The by far largest contribution to the anomalous magnetic moment is of pure QED origin. This is of course the reason why the measurements of $a_e$ and $a_\mu$ until not so long time ago may have been considered as precision tests of QED. This clear dominance of just one type of interaction, the interaction of the charged leptons $e, \mu$ and $\tau$ with the photon, historically, was very important for the development of QFT and QED, since it allowed to test QED as a model theory under very simple, clean and unambiguous conditions. This was very crucial in strengthening our confidence in QFT as a basic theoretical framework. We should remember that it took about 20 years from the invention of QED (Dirac 1928 [$g_e = 2$]) until the first reliable results could be established (Schwinger 1948 [$a_e^{(1)} = \alpha/2\pi$]) after a covariant formulation and renormalization was understood and settled in its main aspects. As precision of experiments improved, the QED part by itself became a big challenge for theorists, because higher order corrections are sizable, and as the order of perturbation theory increases, the complexity of the calculations grow dramatically. Thus experimental tests were able to check QED up to 7 digits in the prediction which requires to evaluate the perturbation expansion up to 5 loops (5 terms in the expansion). The anomalous magnetic moment as a dimensionless quantity exhibits contributions which are just numbers expanded in powers of $\alpha$, what one would get in QED with just one species of leptons, and contributions depending on the mass ratios if different leptons come into play. Thus taking into account all three leptons we obtain functions of the ratios of the lepton masses $m_e, m_\mu$ and $m_\tau$. Considering $a_\mu$, we can cast it into the following form [1, 2]

$$a_\mu^{\text{QED}} = A_1 + A_2(m_\mu/m_e) + A_2(m_\mu/m_\tau) + A_3(m_\mu/m_e, m_\mu/m_\tau)$$

(4.1)

The term $A_1$ in QED is universal for all leptons. Only those internal fermion loops count here where the fermion is the muon (=external lepton). The
contribution $A_2$ has one scale and only shows up if an additional lepton loop of a lepton different from the external one is involved. This requires at least one more loop, thus two at least: for the muon as external lepton we have two possibilities: an additional electron–loop (light–in–heavy) $A_2(m_\mu/m_e)$ or an additional $\tau$–loop (heavy–in–light) $A_2(m_\mu/m_\tau)$ two contributions of quite different character. The first produces large logarithms $\propto \ln(m_\mu/m_e)^2$ and accordingly large effects while the second, because of the decoupling of heavy particles in QED like theories, produces only small effects of order $\propto (m_\mu/m_\tau)^2$.

The two–scale contribution requires a light as well as a heavy extra loops and hence starts at three loop order. We will discuss the different types of contributions in the following. Each of the terms is given in renormalized perturbation theory by an appropriate expansion in $\alpha$:

$$
A_1 = A_1^{(2)} \left( \frac{\alpha}{\pi} \right) + A_1^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + A_1^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A_1^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + A_1^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \cdots
$$

$$
A_2 = A_2^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + A_2^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A_2^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + A_2^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \cdots
$$

$$
A_3 = A_3^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A_3^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + A_3^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \cdots
$$

and later we will denote by

$$
C_L = \sum_{k=1}^{3} A_k^{(2L)},
$$

the total $L$-loop confident of the $(\alpha/\pi)^L$ term.

In collecting various contributions we should always keep in mind the precision of the present experimental result \[3\]

$$
a_\mu^{\text{exp}} = 116592080(63) \times 10^{-11}
$$

and the future prospects of possible improvements \[4\] which could reach an ultimate precision

$$
\delta a_\mu^{\text{fin}} \sim 10 \times 10^{-11}.
$$

(4.2)

For the $n$–loop coefficients multiplying $(\alpha/\pi)^n$ this translates into the required accuracies given in Table \[4.1\]. To match the current accuracy one has to multiply each entry with a factor 6, which is the experimental error in units of $10^{-10}$.

As we will see many contributions are enhancement by large short–distance logarithms of the type $\ln m_\mu/m_e$. These terms are controlled by the RG equation of QED or equivalently by the homogeneous Callan-Symanzik (CS) equation \[5\]

<table>
<thead>
<tr>
<th>$\delta A^{(2)}$</th>
<th>$\delta A^{(4)}$</th>
<th>$\delta A^{(6)}$</th>
<th>$\delta A^{(8)}$</th>
<th>$\delta A^{(10)}$</th>
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<td>$4 \times 10^{-8}$</td>
<td>$1 \times 10^{-5}$</td>
<td>$7 \times 10^{-3}$</td>
<td>3</td>
<td>$1 \times 10^3$</td>
</tr>
</tbody>
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