Points with Type-2 Operations

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Abstract. The algebra of truth values for fuzzy sets of type-2, due to Zadeh, contains as subalgebras those of type-1 and of interval-valued fuzzy sets. It also contains many other interesting subalgebras, some of which could possibly serve as a basis of a useful fuzzy set theory. This paper is about one such subalgebra which we call the subalgebra of points, and which generalizes type-1. We investigate it as an algebra, and determine its automorphism group. In particular, we show that is it a characteristic subalgebra and that its automorphisms are exactly those induced by automorphisms of the containing truth value algebra of fuzzy sets of type-2.

Keywords: Fuzzy truth values, type-2 fuzzy set, type-1 fuzzy set, interval-valued fuzzy set.

1 Introduction

Type-2 fuzzy sets—that is, fuzzy sets with fuzzy sets as truth values were introduced by Zadeh [9] in 1975, and have been the subject of many papers. A basic mathematical treatment is given in [6] and it subalgebras and their automorphisms are the subject of [7,8]. There are several subalgebras of special interest, including copies of the truth value algebra of ordinary type-1 fuzzy and sets and of the truth value algebra of interval-valued fuzzy sets. These are not only subalgebras, but are characteristic in the sense that every automorphism of the algebra induces an automorphism of each of these subalgebras, making them very special subalgebras, further testimony to the "correctness" of Zadeh's generalization. Subalgebras in general are of interest because each could serve as the basis of a fuzzy set theory, where a fuzzy set in this theory is a mapping of a universal set into this subalgebra.

This paper is about one special subalgebra. It is a generalization of the truth value algebra of type-1 fuzzy sets. The elements of the truth value algebra of type-2 fuzzy sets are all functions $[0,1] \to [0,1]$. The elements of the truth value algebra of type-1 fuzzy sets correspond to the characteristic functions of points. That is, they are those functions that are non-zero at exactly one point and have value 1 at that point. The algebra concerned with here is the algebra of those functions which are non-zero at exactly one point but can have any value in

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$(0, 1]$ at that point. This is indeed a subalgebra of the algebra of truth values of type-2 fuzzy sets, and is more general than that of the type-1 truth values. This subalgebra and its automorphisms are principal topics of this paper. It plays a special role in the development of the general theory of the truth value algebra of type-2 fuzzy sets since obviously every mapping $[0, 1] \to [0, 1]$ is the pointwise join of elements of this subalgebra. It is especially important to understand this subalgebra and its automorphisms in determining the automorphisms of the larger algebra.

Elements of the algebra of points will be represented by a pair of points from the unit interval which is reminiscent of interval-valued fuzzy sets. But, as you will see, the operations are not the same, and the algebra of points has quite different properties than the algebra of intervals.

The basic mathematical properties of the truth value algebra of type-2 fuzzy sets are given in [6]. We begin with a review of some relevant definitions.

2 Type-2 Fuzzy Sets

The algebra of truth values for fuzzy sets of type-2 is the set of all mappings of $[0, 1]$ into $[0, 1]$ with operations certain convolutions of operations on $[0, 1]$. These operations are as follows.

Definition 1. *The* ***algebra of truth values for type-2 fuzzy sets*** *is the algebra*

$$\mathbb{M} = (\,[0,1]^{[0,1]}\,, \sqcup, \sqcap, {}^*, \bar{0}, \bar{1}) \tag{1}$$

where the operations are defined by

1. $(f \sqcup g)(x) = \sup \{f(y) \wedge g(z) : y \vee z = x\}$
2. $(f \sqcap g)(x) = \sup \{f(y) \wedge g(z) : y \wedge z = x\}$
3. $f^*(x) = \sup \{f(y) : 1 - y = x\}$
4. $\bar{1}(x) = 1$ *if* $x = 1$ *and* $\bar{1}(x) = 0$ *if* $x \neq 1$
5. $\bar{0}(x) = 1$ *if* $x = 0$ *and* $\bar{0}(x) = 0$ *if* $x \neq 0$

Note that $f^*(x) = f(1-x)$. A **fuzzy subset of type-2 of a set** S is a mapping $f : S \to [0, 1]^{[0,1]}$, and operations on the set $\mathcal{F}_2(S)$ of all such fuzzy subsets are given pointwise from the operations in $\mathbb{M}$. Thus we have the **algebra** $\mathcal{F}_2(S) = (Map(S, [0, 1]^{[0,1]}), \sqcup, \sqcap, {}^*, \bar{0}, \bar{1})$ **of fuzzy subsets of type-2 of the set** S**.** The same equations hold in $\mathcal{F}_2(S)$ as in $\mathbb{M}$**.**

Determining the properties of the algebra $\mathbb{M}$ is a bit tedious, but is helped by introducing the following auxiliary operations.

Definition 2. *For* $f \in \mathbb{M}$*, let* f^L *and* f^R *be the elements of* $\mathbb{M}$ *defined by*

$$f^L(x) = \sup \{f(y) : y \leq x\}$$
$$f^R(x) = \sup \{f(y) : y \geq x\}$$