Edge Fault-Diameter of Cartesian Product of Graphs

Iztok Banič¹,² and Janez Žerovnik¹,²

¹ University of Maribor, Smetanova 17, Maribor 2000, Slovenia
² Institute of Mathematics, Physics and Mechanics, Jadranska 19, Ljubljana
iztok.banic@uni-mb.si, janez.zerovnik@imfm.uni-lj.si

Abstract. Let $G$ be a $k_G$-edge connected graph and $\overline{D}_c(G)$ denote the diameter of $G$ after deleting any of its $c < k_G$ edges. We prove that if $G_1, G_2, \ldots, G_q$ are $k_1$-edge connected, $k_2$-edge connected, $\ldots$, $k_q$-edge connected graphs and $0 \leq a_1 < k_1$, $0 \leq a_2 < k_2, \ldots, 0 \leq a_q < k_q$ and $a = a_1 + a_2 + \ldots + a_q + (q - 1)$, then the edge fault-diameter of $G$, the Cartesian product of $G_1, G_2, \ldots, G_q$, with $a$ faulty edges is $\overline{D}_a(G) \leq \overline{D}_{a_1}(G_1) + \overline{D}_{a_2}(G_2) + \ldots + \overline{D}_{a_q}(G_q) + 1$.

Keywords: Cartesian graph products, edge fault-diameter, interconnection network.

1 Introduction

In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and the delays in communication must not be too long. Extensively studied network topologies in this context include graph products and bundles. For example the meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some well-known topologies are Cartesian graph bundles, i.e. some twisted hypercubes [5,8] and multiplicative circulant graphs [16]. Other graph products, sometimes under different names, have been studied as interesting communication network topologies [16,15,4].

Furthermore, an interconnection network should be fault-tolerant. Since nodes and edges of a network do not always work, if some nodes or edges are faulty, some information may not be transmitted through these nodes and by these edges. Usually, it is assumed that either only nodes or only edges are faulty and hence either node fault-diameter (or, simply, fault diameter) or edge fault-diameter is studied. The fault diameter has been determined for many important networks recently [7,14,18]. The concept of fault diameter of Cartesian product graphs was first described in [13], but the upper bound was wrong, as shown by Xu, Xu and Hou who corrected the mistake [18]. An upper bound for the fault diameter of Cartesian graph bundles was given in [1] and for arbitrary

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Cartesian products in [2]. When a preliminary version of [112] was presented at the conference [3], a question was asked whether similar results can be proved for the edge fault-diameter.

In this paper we study the diameter of a graph after deleting some of the edges. As a $k$-edge connected graph remains connected if up to $k - 1$ edges are missing, we study the diameter of a graph with any permitted number of edges deleted. In this paper, we prove the three theorems listed below.

**Theorem 1.** Let $G_1, G_2, \ldots, G_q$ be $k_1$-edge connected, $k_2$-edge connected, \ldots, $k_q$-edge connected graphs. Let $0 \leq a_1 < k_1, 0 \leq a_2 < k_2, \ldots, 0 \leq a_q < k_q$ and $a = a_1 + a_2 + \ldots + a_q + (q - 1)$. Then the edge fault-diameter of a graph $G$, the Cartesian product of $G_1, G_2, \ldots, G_q$, with a faulty edges is

$$\overline{D}_a(G) \leq \overline{D}_{a_1}(G_1) + \overline{D}_{a_2}(G_2) + \ldots + \overline{D}_{a_q}(G_q) + 1.$$ 

In fact, Theorem 1 implies a more precise result for an upper bound of the edge fault-diameter of a Cartesian product of graphs:

**Theorem 2.** Let $G_1, G_2, \ldots, G_q$ be $k_1$-edge connected, $k_2$-edge connected, \ldots, $k_q$-edge connected graphs, and $G$ the Cartesian product of $G_1, G_2, \ldots, G_q$. Let $0 \leq a < k_1 + k_2 + \ldots + k_q$. Then

$$\overline{D}_a(G) \leq \min\{\overline{D}_{a_1}(G_1) + \overline{D}_{a_2}(G_2) + \ldots + \overline{D}_{a_q}(G_q) + 1 \mid a_1 + a_2 + \ldots + a_q = a - (q - 1), 0 \leq a_1 < k_1, 0 \leq a_2 < k_2, \ldots, 0 \leq a_q < k_q\}.$$ 

Furthermore, for cases with a small number of faulty edges we prove the exact formula for computing the edge fault-diameter:

**Theorem 3.** Let $G_1, G_2, \ldots, G_q$ be connected graphs, and $G = \square_{i=1}^q G_i$. Then

1. $\overline{D}_a(G) = \sum_{i=1}^q \overline{D}_0(G_i) = \overline{D}_0(G)$ for $0 \leq a < q - 1$;
2. $\overline{D}_0(G) \leq \overline{D}_{q-1}(G) \leq \overline{D}_0(G) + 1$.

Note that $\overline{D}_0(G)$ is just the diameter of $G$. In fact we prove more than stated in Theorem 3, namely: if all the factors are trees and there is a factor that is a complete graph (i.e. $K_2$), then $\overline{D}_{q-1}(G) = \overline{D}_0(G) + 1$; and if none of the factors is a complete graph, then $\overline{D}_{q-1}(G) = \overline{D}_0(G)$.

While the results proven here are very similar to the results for the node fault version [2] and the methods used are similar, we were not able to find a faster proof, for example a theorem which would “translate” the results from node fault version to the edge fault version of the theorems. It may be an interesting task to look for such a transformation or to find reasons why this seemingly is difficult. On the positive side, we believe that the same method can be used to prove the edge fault version of the result [1] for graph bundles. Another interesting question is whether the approach can be extended to prove analogous bounds for the mixed problem, in which both nodes and edges may be faulty.