1 Introduction

Differential equations are types of equations that arise from the mathematical modelling, or simulation, of physical phenomena or from engineering applications; for example: the flow of water, the decay of radioactive substances, bodies in motion, electrical circuits, chemical processes, etc. When we cannot solve differential equations analytically, we must resort to numerical methods. Unfortunately, numerical methods do not give us an exact solution, and an amount of error is introduced in the answer. It is our goal to utilize concepts from Control Theory in order to minimize this error. For our study, we shall focus on ordinary differential equations (ODEs).

Without going into details, we state that ODEs may be solved analytically by methods such as: substitution, by using an integrating factor, or by separation of variables, to name a few. However, there are times when the differential equation is too complicated to be solved analytically; it is then when we need to use numerical methods to obtain an approximation to the solution.

Numerical methods may be single-step, in which case, in order to calculate the next point of the solution function of the differential equation, it is only necessary to have information of the preceding point. Examples are Euler’s method, Taylor’s method, or Runge-Kutta method. Or they can be multi-step, where information of at least two preceding points is necessary in order to calculate the next point of the solution function. Examples are Adams-Bashforth method or Adams-Moulton method [3].

As mentioned earlier, numerical methods can only give us an approximation of the solution, and the difference with the exact solution is referred to as the error. This error is of varying magnitude for each method. It is the motivation of this work to apply robust and optimal control concepts to these methods to reduce the error of the solution [1]; in contrast to other control theoretical techniques which look to regulate the stepsize selection in the solution of the ordinary differential equation [4]. Figure 1 is a block diagram of the complete ordinary differential equation solver.

The Plant consists of the Predictor and Corrector blocks. The Predictor generates an approximation to the solution of the ODE, while the Corrector improves this approximation. The Model contains the exact solution of the ODE for comparison purposes, from which an error is generated from the difference between the approximation and
the exact solution. Our work will focus on the error term $e_{n+1}$ and the design of the Regulator, whose function is to minimize the error further. For the statistical analysis, we will focus on the study of error distribution for the four-step Adams-Bashforth method. We develop a model with a term which has a mean zero and a constant variance. Then we will compare these two models developed by control theory and by time series theory.

2 A Simple Example

In order to illustrate the technique. We shall begin with a simple test ODE,

$$\frac{dx}{dt} = \lambda x, \quad 0 \leq t \leq 1, \quad y(0) = 1,$$

with its well known solution $x(t) = e^{\lambda t}$ and we shall use Euler’s method for the numerical computation of this solution. Euler’s method is not sufficiently accurate to be used in real applications but it is simple enough for illustration purposes.

The Euler algorithm is given by

$$x_{n+1} = hf(t_n, x_n) \quad \text{for} \quad n = 0, 1, \ldots, N - 1,$$

where $N$ is a positive integer and the approximate value of $x$, $x_n$, is found at given points $t_n$ (called mesh points),

$$t_n = a + nh \quad \text{for} \quad n = 0, 1, \ldots, N,$$

where $h$ is the step size given by $h = \frac{b - a}{N} = t_{n+1} - t_n$, on the interval $[a, b]$.

Thus we have,

$$x_{n+1} = x_n + \lambda hx_n \quad (1)$$

$$x_{n+1} = (1 + \lambda h)x_n \quad (2)$$

And the error can be found from

$$e_n = x_n - e^{\lambda h n} \quad (3)$$