4 Mathematical Description of Infinitely Extended Quantum Systems

From the discussion of the previous Chapter, it appears that the description of infinite systems looks much more difficult than in the finite dimensional case, above all because of the existence of (too) many possible representations of the algebra of canonical variables. A big step in the direction of controlling the problem has been taken by Haag et al., who emphasized the need of exploiting crucial physical properties of the algebra of observables in order to restrict their possible representations to the physically relevant ones. The crucial ingredient is the localization property of observable operations.

4.1 Local Structure

Any physically realizable operation is necessarily localized in space, since we cannot perform measurements or act on the system over the whole space. In order to encode this property in the structure of the algebra of observables, it is convenient to view it as generated by canonical variables or observables which have localization properties. Thus, for each bounded space region $V$, one has the $C^*$-algebra $\mathcal{A}(V)$ of all observables (or canonical variables) localized in $V$.

A concrete realization of such a structure is obtained by considering canonical variables which have localization properties in the sense of (3.3). For regular test functions $f, g$ of compact support contained in $V$, (typically $f, g \in \mathcal{D}(V)$), one considers the set of localized canonical variables

$$a(f) \equiv \int dx \, \psi(x) \, \bar{f}(x), \quad a^*(g) = \int dx \, \psi^*(x) \, g(x),$$

where $\psi(x)$ is a field “strictly localized in $x$” (see (3.3)). The algebra generated by such variables can be taken as the Heisenberg algebra localized in $V$. Similarly, a Weyl algebra localized in $V$ is generated by the exponentials of

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71 For a general discussion of this strategy, see R. Haag, *Local Quantum Physics*, Springer 1996.


DOI 10.1007/978-3-540-73593-9_4 © Springer-Verlag Berlin Heidelberg 2008
the above localized canonical variables

\[ U(f) = \exp \left[ i (a(f) + a(f)^*) \right], \quad V(g) = \exp \left[ a(g) - a(g)^* \right]. \]

Quite generally, the association \( V \to \mathcal{A}(V) \) realizes the identification of the algebras of observables localized in the volume \( V \) as \( V \) varies. The consistency of the physical interpretation requires that such a mapping satisfies the so-called *isotony* property, namely \( \mathcal{A}(V_1) \subseteq \mathcal{A}(V_2) \), whenever \( V_1 \subseteq V_2 \).

The physically motivated concept of localization has an algebraic translation in terms of commutation relations. For (equal time) space localization, the local structure of the algebras \( \mathcal{A}(V) \) is formalized by the property

\[ [\mathcal{A}(V), \mathcal{A}(V')] = 0, \quad \text{if} \quad V \cap V' = \emptyset. \quad (4.1) \]

For relativistic systems, it is more convenient to introduce algebras localized in bounded (open) space time regions \( \mathcal{O} \) (usually taken as causally complete as it is the case of the diamonds or double cones\(^{72}\)). Then, the locality property reads

\[ [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0, \quad (4.2) \]

whenever \( \mathcal{O}_2 \) is spacelike with respect to \( \mathcal{O}_1 \), briefly \( \mathcal{O}_2 \subseteq \mathcal{O}_1' \equiv \text{the causal complement of } \mathcal{O}_1 \). For observable algebras, this is the mathematical formulation of Einstein causality.\(^{73}\)

The union of all \( \mathcal{A}(V) \) (or \( \mathcal{A}(\mathcal{O}) \)) is called the *local algebra*

\[ \mathcal{A}_L \equiv \bigcup_V \mathcal{A}(V), \quad V = V, \quad \text{or} = \mathcal{O}. \quad (4.3) \]

We have already argued before that it is convenient (if not necessary) to have a \( C^* \)-algebra and therefore one has to complete \( \mathcal{A}_L \). As we shall see, this is a delicate point having deep connections with the dynamics and the physical description of the system. The most natural and simple choice is to consider the norm closure

\[ \mathcal{A} \equiv \overline{\mathcal{A}_L}. \quad (4.4) \]

The norm closure leads to the smallest \( C^* \)-algebra generated by strictly local elements, all other topologies, like the (ultra-)strong and the (ultra-)weak being weaker, and therefore it gives the \( C^* \)-algebra with best localization properties. For this reason, the norm closure \( \mathcal{A} \) is called the *quasi local algebra*.

Since the time evolution is one of the possible physically realizable operations, in order to have a consistent physical picture, the algebra of observables, 

\(^{72}\) A set \( \mathcal{O} \) of points is *causally complete* if it coincides with its double causal complement, i.e. if \( \mathcal{O} = (\mathcal{O}')' \), where \( \mathcal{O}' \) (called the *causal complement of \( \mathcal{O} \)) denotes the set of all points which are spacelike with respect to all points of \( \mathcal{O} \).